

ALGEBRAIC HAMILTONIAN ACTIONS

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ABSTRACT. In this paper we deal with a Hamiltonian action of a reductive algebraic group G on an irreducible normal affine Poisson variety X . We study the quotient morphism $\mu_{G,X}/G : X//G \rightarrow \mathfrak{g}/G$ of the moment map $\mu_{G,X} : X \rightarrow \mathfrak{g}$. We prove that for a wide class of Hamiltonian actions (including, for example, actions on generically symplectic varieties) all fibers of the morphism $\mu_{G,X}/G$ have the same dimension. We also study the "Stein factorization" of $\mu_{G,X}/G$. Namely, let $C_{G,X}$ denote the spectrum of the integral closure of $\mu_{G,X}^*(\mathbb{K}[\mathfrak{g}]^G)$ in $\mathbb{K}(X)^G$. We investigate the structure of the \mathfrak{g}/G -scheme $C_{G,X}$. Our results partially generalize those obtained by F. Knop for the actions on cotangent bundles and symplectic vector spaces.

1. INTRODUCTION

1.1. Main objects of study. In this paper we study Hamiltonian actions of reductive algebraic groups on Poisson varieties. The ground field \mathbb{K} is algebraically closed and of characteristic 0.

A Poisson variety X is a variety whose structure sheaf is a sheaf of Poisson algebras. Poisson morphisms of Poisson varieties are defined in an obvious way. Let G be a reductive group acting on X by Poisson automorphisms. The action $G : X$ is said to be *Hamiltonian* if it is equipped with a G -equivariant linear map $\mathfrak{g} \rightarrow \mathbb{K}[X], \xi \mapsto H_\xi$, where \mathfrak{g} denotes the Lie algebra of G , such that the derivation $\{H_\xi, \cdot\}$ of the algebra $\mathbb{K}(X)$ coincides with the velocity vector field ξ_* . Under these conditions, the corresponding homomorphism $S(\mathfrak{g}) \rightarrow \mathbb{K}[X]$ is a homomorphism of Poisson algebras. A Poisson variety equipped with a Hamiltonian action of G is said to be a Hamiltonian G -variety.

The morphism $\mu_{G,X} : X \rightarrow \mathfrak{g}^*$ defined by $\langle \mu_{G,X}(x), \xi \rangle = H_\xi(x), x \in X, \xi \in \mathfrak{g}$, is called the *moment map*. Since G is reductive, the algebra \mathfrak{g} possesses a nondegenerate symmetric bilinear G -invariant form (\cdot, \cdot) . We may assume additionally that this form is nondegenerate on any Lie algebra of a reductive subgroup of G . Fix such a form and identify \mathfrak{g}^* and \mathfrak{g} . So we can consider $\mu_{G,X}$ as a morphism $X \rightarrow \mathfrak{g}$. We also consider a morphism $\psi_{G,X} : X \rightarrow \mathfrak{g}/G$, the composition of $\mu_{G,X}$ and the quotient morphism $\pi_{G,\mathfrak{g}} : \mathfrak{g} \rightarrow \mathfrak{g}/G$.

It is interesting to study a kind of "Stein factorization" for the morphism $\psi_{G,X}$. Let X be a normal irreducible Hamiltonian G -variety. Denote the integral closure of $\psi_{G,X}^*(\mathbb{K}[\mathfrak{g}]^G)$ in $\mathbb{K}(X)^G$ by A . This is a finitely-generated subalgebra of $\mathbb{K}[X]^G$. Put $C_{G,X} = \text{Spec}(A)$. This is a normal irreducible affine variety. There are two natural morphisms: the G -invariant dominant morphism $\tilde{\psi}_{G,X} : X \rightarrow C_{G,X}$ and the finite morphism $\tau_{G,X} : C_{G,X} \rightarrow \mathfrak{g}/G$. We remark that, by the construction of $C_{G,X}$, G permutes transitively connected components

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of a general fiber of $\tilde{\psi}_{G,X}$. The variety $C_{G,X}$ and the morphisms $\tilde{\psi}_{G,X} : X \rightarrow C_{G,X}$ and $\tau_{G,X} : C_{G,X} \rightarrow \mathfrak{g}//G$ are the main objects of study in this paper.

One of motivations for this study comes from the theory of Hamiltonian actions of compact Lie groups on smooth symplectic manifolds. Namely, let K be a connected compact Lie group and X a symplectic K -manifold. As above, one can define the moment map $\mu : X \rightarrow \mathfrak{k}$. Choose a Weyl chamber $C \subset \mathfrak{k}$ and consider the continuous map $\psi : X \rightarrow C$ mapping x to $K\mu(x) \cap C$. This is an analog of $\psi_{G,X}$ in this situation. The map ψ has the following properties (see [Ki]):

- (1) The image of ψ is a convex polytope.
- (2) All fibers of ψ are connected.

However, general fibers of $\psi_{G,X}$ are, in general, not connected even for connected G . The action of $G = \mathrm{SL}_2$ on $\mathbb{K}^2 \oplus \mathbb{K}^2$ provides an example. Therefore it seems that the right analog of ψ in the algebraic situation is the morphism $\tilde{\psi}_{G,X}$. We will see in the sequel that the image of $\tilde{\psi}_{G,X}$ possesses nice properties (at least for sufficiently good, for example, generically symplectic, affine varieties X). We also conjecture that for such varieties X all fibers of $\tilde{\psi}_{G,X} // G : X // G \rightarrow C_{G,X}$ are irreducible¹. This would imply the connectedness property for all fibers of $\tilde{\psi}_{G,X}$.

The second motivation of our study comes from Invariant theory. It turns out that the subalgebra $\mathbb{K}[C_{G,X}] \subset \mathbb{K}[X]^G$ is closely related to the subalgebra of all functions lying in the center of the Poisson algebra $\mathbb{K}(X)^G$. In particular, if $\mathbb{K}(X)^G$ is commutative, then $C_{G,X}$ is closely related to $X // G$.

The idea to study $C_{G,X}, \tilde{\psi}_{G,X}, \tau_{G,X}$ belongs to F. Knop. In [Kn1] he showed that if G is connected, X_0 is a smooth irreducible G -variety and $X = T^*X_0$, then $C_{G,X}$ is an affine space and the morphism $\psi_{G,X}$ is equidimensional. He also described the morphism $\tau_{G,X}$. Namely, there is a subspace \mathfrak{a}_{G,X_0} in a Cartan subalgebra \mathfrak{t} of \mathfrak{g} and a subgroup W_{G,X_0} in the quotient $N_G(\mathfrak{a}_{G,X_0})/Z_G(\mathfrak{a}_{G,X_0})$ such that $C_{G,X} \cong \mathfrak{a}_{G,X_0}/W_{G,X_0}$ and $\tau_{G,X}$ is the morphism $\mathfrak{a}_{G,X_0}/W_{G,X_0} \rightarrow \mathfrak{g}//G$ induced by the restriction of functions from \mathfrak{g} to \mathfrak{a}_{G,X_0} . Since $C_{G,X}$ is an affine space, the group W_{G,X_0} is generated by reflections. The subalgebra $\mathbb{K}[C_{G,X}]$ coincides with the center of the Poisson algebra $\mathbb{K}[X]^G$. Recently, F. Knop obtained the analogous results for linear Hamiltonian actions, see [Kn5]. Moreover, in the case of cotangent bundles all fibers of $\tilde{\psi}_{G,X}$ are connected, see [Kn4].

In the case when X_0 is a smooth irreducible G -variety the group W_{G,X_0} is an important birational invariant of X . In a subsequent paper we will apply some results and constructions of the present paper to the problem of the computation of W_{G,X_0} .

1.2. Statement of the results. We need some definitions.

Definition 1.2.1. Let G be an arbitrary algebraic group. A G -variety X is called *G -irreducible* if G acts transitively on the set of irreducible components of X .

X is G -irreducible iff $\mathbb{K}(X)^G$ is a field.

Next, we define important numerical invariants of a Hamiltonian G -variety. For a G -variety X let $m_G(X)$ denote the maximal dimension of a G -orbit on X .

Definition 1.2.2. Let X be a G -irreducible Hamiltonian G -variety. The *rank* of X is, by definition, the number $\mathrm{rk}_G(X) = m_G(\overline{\mathrm{im} \mu_{G,X}})$. The difference $m_G(X) - \mathrm{rk}_G(X)$ is called the

¹After this paper was submitted Friedrich Knop found a counterexample to this conjecture

lower defect of X and is denoted by $\underline{\text{def}}_G(X)$. The upper defect is the dimension of $\overline{\text{im } \psi_{G,X}}$, it is denoted by $\overline{\text{def}}_G(X)$.

Theorem 1.2.3. *Suppose X is an affine G -irreducible Hamiltonian G -variety. Then the codimension of any fiber of $\mu_{G,X} // G : X // G \rightarrow \mathfrak{g} // G$ in $X // G$ is not less than $\underline{\text{def}}_G(X)$.*

Definition 1.2.4. A G -irreducible Hamiltonian G -variety X is called *equidefectinal* if $\underline{\text{def}}_G(X) = \overline{\text{def}}_G(X)$. In this case we call $\underline{\text{def}}_G(X) = \overline{\text{def}}_G(X)$ the defect of X and denote it by $\text{def}_G(X)$.

We will see in Subsection 3.4 that if X is generically symplectic (Definition 2.2.4) or $m_G(X) = \dim G$, then X is equidefectinal.

If X is equidefectinal, then all fibers of $\mu_{G,X} // G$ have the same dimension. It is not clear whether $\psi_{G,X}$ possesses this property. However, it is so when X is smooth and symplectic. We will prove this in a subsequent paper [Lo]. Moreover, it can be shown that any fiber of $\psi_{G,X}$ has a component of the "right" dimension.

The next two theorems describe the morphism $\tau_{G,X} : C_{G,X} \rightarrow \mathfrak{g} // G$. To state them we introduce some factorization $\tau_{G,X} = \tau_{G,X}^1 \circ \tau_{G,X}^2$.

Let X be a normal irreducible equidefectinal Hamiltonian G -variety. For a point $x \in X$ in general position we put $L = Z_{G^\circ}(\xi_s)$, where $\xi = \mu_{G,X}(x)$. Note that L is defined uniquely up to G° -conjugacy. Put $\mathcal{P}^r = \{\xi \in \mathfrak{l}(\mathfrak{z}_\mathfrak{g}(\xi_s)) \subset \mathfrak{l}\}$. It follows from results of Subsections 5.1, 5.2, that $\mu_{G,X}^{-1}(\mathcal{P}^r)$ is a normal $N_G(L)$ -irreducible variety. Choose a component $Y \subset \mu_{G,X}^{-1}(\mathcal{P}^r)$. Later on we will see that the closure of the image of the projection $p : \mu_{G,X}(Y) \rightarrow \mathfrak{l} / [\mathfrak{l}, \mathfrak{l}] \cong \mathfrak{z}(\mathfrak{l})$ in \mathfrak{g} is an affine subspace in $\mathfrak{z}(\mathfrak{l})$ of dimension $\text{def}_G(X)$ (Proposition 4.4.1, Remark 5.2.3). We denote this subspace by $\mathfrak{a}_{G,X}^{(Y)}$. The group $W_{G,X}^{(Y)} = N_G(L, Y) / L$ acts on $\mathfrak{a}_{G,X}^{(Y)}$ by affine transformations. It turns out (see Subsection 5.2) that $\tau_{G,X} = \tau_{G,X}^1 \circ \tau_{G,X}^2$, where $\tau_{G,X}^2 : C_{G,X} \rightarrow \mathfrak{a}_{G,X}^{(Y)} / W_{G,X}^{(Y)}$ is a finite dominant morphism and $\tau_{G,X}^1 : \mathfrak{a}_{G,X}^{(Y)} / W_{G,X}^{(Y)} \rightarrow \mathfrak{g} // G$ is the finite morphism corresponding to the restriction of functions from $\mathbb{K}[\mathfrak{g}]^G$ to $\mathfrak{a}_{G,X}^{(Y)}$.

In the case when X_0 is a quasi-affine algebraic variety, $X = T^*X_0$ and G is connected the pair $(\mathfrak{a}_{G,X}^{(Y)}, W_{G,X}^{(Y)})$ is G -conjugate to the pair $(\mathfrak{a}_{G,X_0}, W_{G,X_0})$ established by Knop. Our construction of $\mathfrak{a}_{G,X}^{(Y)}, W_{G,X}^{(Y)}$ is inspired by Vinberg's variant of the definition of $\mathfrak{a}_{G,X_0}, W_{G,X_0}$ (see [V2]). Note that Vinberg's construction is implicitly contained in [Kn2]. The analogous construction of $\mathfrak{a}_{G,X_0}, W_{G,X_0}$ for general X_0 was obtained in [T].

Theorem 1.2.5. *Let X be an equidefectinal normal irreducible affine Hamiltonian G -variety. Then the morphism $\tilde{\psi}_{G,X} // G : X // G \rightarrow C_{G,X}$ is open. In particular, $\text{im } \tilde{\psi}_{G,X} = \text{im}(\tilde{\psi}_{G,X} // G)$ is an open subset of $C_{G,X}$. Moreover, there is a normal $W_{G,X}^{(Y)}$ -variety Z such that $C_{G,X} \cong Z / W_{G,X}^{(Y)}$ and a finite morphism $\tau : Z \rightarrow \mathfrak{a}_{G,X}^{(Y)}$ such that:*

- (1) $\tau_{G,X}^2 = \tau / W_{G,X}^{(Y)}$.
- (2) The morphism τ is étale in all points of $\pi_{W_{G,X}^{(Y)}, Z}^{-1}(\text{im } \tilde{\psi}_{G,X})$.

In particular, we get a partial description of singularities of $\text{im } \tilde{\psi}_{G,X}$. For Z we take the variety $C_{L,R}$, where R is an affine normal irreducible equidefectinal Hamiltonian L -variety constructed from X (the P_u -reduction of X , see below).

Under some additional restriction on the action $G : X$ a more precise statement can be obtained. The restriction is a presence of some "good" action of the one-dimensional torus \mathbb{K}^\times on X . Let us give the precise definition.

Definition 1.2.6. An affine Hamiltonian G -variety X equipped with an action $\mathbb{K}^\times : X$ commuting with the action of G is said to be *conical* if the following two conditions are fulfilled

- (Con1) The morphism $\mathbb{K}^\times \times X//G \rightarrow X//G$ induced by the action $\mathbb{K}^\times : X$ can be extended to a morphism $\mathbb{K} \times X//G \rightarrow X//G$.
- (Con2) There exists a positive integer k such that $\mu_{G,X}(tx) = t^k \mu_{G,X}(x)$ for all $t \in \mathbb{K}^\times, x \in X$. An integer k satisfying the assumptions of (Con2) is called the *degree* of X .

For example, cotangent bundles and symplectic vector spaces with the natural actions of \mathbb{K}^\times are conical (see Subsection 3.3).

Theorem 1.2.7. *Let X be a conical Hamiltonian G -variety satisfying the assumptions of Theorem 1.2.5. Then $\mathfrak{a}_{G,X}^{(Y)}$ is a subspace in $\mathfrak{z}(\mathfrak{l})$ and $\tau_{G,X}^1$ is an isomorphism. If, in addition, X is generically symplectic, then $\mathbb{K}[C_{G,X}]$ coincides with the subalgebra of all regular G -invariants lying in the center of the Poisson field $\mathbb{K}(X)^G$.*

Under the assumptions of the previous theorem, $\mathbb{K}[C_{G,X}]$ coincides with the center of $\mathbb{K}[X]^G$ provided $\mathbb{K}(X)^G = \text{Quot}(\mathbb{K}[X]^G)$. It turns out that the latter is true under some additional assumptions.

Definition 1.2.8. A Hamiltonian G -variety X is called *strongly equidefectinal* if there exists a stratification $X = \coprod_i X_i$ by locally closed G -irreducible equidefectinal Hamiltonian subvarieties (see Definition 3.1.3) $X_i \subset X$ such that $X_i = X_i^{max}$.

Some classes of strongly equidefectinal Hamiltonian varieties are listed in Subsection 3.4.

Theorem 1.2.9. *Suppose X is a strongly equidefectinal normal affine irreducible Hamiltonian G -variety. Then*

- (1) *A fiber of $\pi_{G,X}$ in general position contains a dense G -orbit or, equivalently, $\mathbb{K}(X)^G = \text{Quot}(\mathbb{K}[X]^G)$.*
- (2) *The following conditions are equivalent:*
 - (a) *The action $G : X$ is stable, i.e. a fiber of $\pi_{G,X}$ in general position consists of one orbit.*
 - (b) *The stabilizer in general position for the action $G : X$ exists and is reductive.*
 - (c) *The subset of $\overline{\text{im}} \mu_{G,X}$ consisting of semisimple elements is dense in $\overline{\text{im}} \mu_{G,X}$.*

1.3. Some key ideas. There are three main ingredients of the proofs. Let us give their short (and not very precise) descriptions.

The first ingredient is the structure theory of a special class of Hamiltonian G -varieties, namely central-nilpotent ones.

Definition 1.3.1. A Hamiltonian G -variety X is called *central-nilpotent* (or, shortly, CN) if $\mu_{G,X}(x)_s \in \mathfrak{z}(\mathfrak{g})$ for any $x \in X$.

It is not very difficult to prove Theorems 1.2.3, 1.2.5 in the CN case. Furthermore, irreducible normal affine CN Hamiltonian G -varieties have a nice description provided G is connected. Let us state this result.

There are two important classes of affine CN Hamiltonian G -varieties. Firstly, one can consider a Hamiltonian $G/(G, G)$ -variety X_0 as a Hamiltonian G -variety. Such Hamiltonian G -varieties are clearly CN. To obtain one more example, consider a nilpotent element $\eta \in \mathfrak{g}$. By Example 3.2.7, $X_1 := \text{Spec}(\mathbb{K}[G/(G_\eta)^\circ])$ is a Hamiltonian G -variety. This variety is

again CN. Thus the product $X_0 \times X_1$ is CN. Consider a finite group Γ acting on $X_0 \times X_1$ by Poisson automorphisms preserving the moment map. We get a CN Hamiltonian variety $X_0 \times X_1/\Gamma$. It turns out that any affine irreducible normal CN Hamiltonian G -variety has such a form provided $G \cong Z(G)^\circ \times (G, G)$. The last requirement is not restrictive because any connected reductive algebraic group possesses a covering satisfying this requirement. Using this classification it is not very difficult to prove Theorem 1.2.9.

The second ingredient is the local theory of Hamiltonian actions on quasi-projective varieties based on the Guillemin and Sternberg local cross-section theorem (see [GS],[Kn3]). Roughly speaking, the theorem reduces the study of a Hamiltonian G -variety in an étale neighborhood of a point $x \in X$ to the study of a Hamiltonian action of the Levi subgroup $Z_{G^\circ}(\mu_{G,X}(x)_s)$ on some locally closed subvariety of X . This subvariety is called a *cross-section*. An example of a cross-section is the Hamiltonian L -variety $Y \subset \mu_{G,X}^{-1}(\mathfrak{l}^{pr})$ mentioned above. Using local cross-sections we complete the proof of Theorem 1.2.9.

To describe the third ingredient suppose that X is an irreducible normal affine equidimensional Hamiltonian G -variety. Roughly speaking, Y is a CN Hamiltonian variety "approximating" X . However, there is another CN Hamiltonian L -variety approximating X even better. This variety is constructed from Y and an appropriate parabolic subgroup $P \subset G$ with Levi subgroup L and is called the *P_u -reduction of X associated with Y* . The name is chosen because our construction is, in some sense, a modification of the Marsden-Weinstein reduction, see [MW]. The idea is as follows. We want to consider the Marsden-Weinstein reduction for the action $P_u : X$, that is, the quotient $\mu_{G,X}^{-1}(\mathfrak{p})//P_u$. However, this quotient, in general, seems to be very bad, possibly, it is not even a variety. Therefore we take a "good" component of $\mu_{G,X}^{-1}(\mathfrak{p})$, namely $Z = \overline{P_u Y}$, and consider not the whole algebra $\mathbb{K}[Z]^{P_u}$ but its subalgebra A_Z generated by $H_\xi|_Z, \xi \in \mathfrak{l}$, and $f|_Z, f \in \mathbb{K}[X]^{P_u}$. It turns out that this subalgebra is finitely generated. The P_u -reduction R is the normalization of the spectrum of the subalgebra. It is that variety mentioned after Theorem 1.2.5. R possesses the natural structure of a Hamiltonian L -variety (the hamiltonians are $H_\xi|_{\overline{P_u Y}}, \xi \in \mathfrak{l}$). The P_u -reduction is used to reduce the proofs of Theorems 1.2.3, 1.2.5 to the CN case.

2. POISSON VARIETIES

In Subsection 2.1 we define a Poisson (not necessarily smooth) variety. In Subsection 2.2 we define the Poisson bivector of a Poisson variety and study its properties. In Subsection 2.3 main examples of Poisson varieties are given. In Subsection 2.4 we introduce a stratification of a Poisson variety by smooth Poisson subvarieties with the Poisson bivector of constant rank. Almost all definitions and results of this section are well-known in the symplectic case.

2.1. The main definition. A commutative associative algebra A with unit is called *Poisson* if it is equipped with a skew-symmetric bilinear bracket $\{\cdot, \cdot\} : A \otimes A \rightarrow A$ satisfying the Leibnitz and Jacobi identities, that is

$$(2.1) \quad \{f, gh\} = \{f, g\}h + \{f, h\}g, \forall f, g, h \in A,$$

$$(2.2) \quad \{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0, \forall f, g, h \in A.$$

Thus the map $f \mapsto \{f, g\}$ is a derivation of A for any $g \in A$.

Poisson homomorphisms of Poisson algebras are defined in a natural way. An ideal $I \subset A$ is called Poisson if $\{A, I\} \subset I$. For such an ideal I the algebra A/I possesses a unique Poisson bracket such that the projection $A \rightarrow A/I$ is a Poisson homomorphism.

Proposition 2.1.1. *Let $A \subset B$ be an algebraic extension of integral domains. Suppose A is equipped with a Poisson bracket $\{\cdot, \cdot\}_A$ and B with some bracket $\{\cdot, \cdot\}_B$ (here a bracket is a skew-symmetric bilinear operation satisfying the Leibnitz identity) such that*

$$(2.3) \quad \{f, g\}_B = \{f, g\}_A, \forall f, g \in A.$$

Then $\{\cdot, \cdot\}_B$ is a Poisson bracket. If brackets $\{\cdot, \cdot\}_B^1, \{\cdot, \cdot\}_B^2$ on B satisfy (2.3), then they coincide.

Proof. This is a consequence of the fact that $\{\cdot, \cdot\}$ is a biderivation and the uniqueness of a lifting of a derivation for algebraic extensions of integral domains (see [Le], Chapter 10). \square

Definition 2.1.2. A variety X is called *Poisson*, if its structure sheaf is a sheaf of Poisson algebras. A subvariety $Y \subset X$ is called Poisson if its ideal sheaf is a sheaf of Poisson ideals. A morphism of Poisson varieties is called Poisson if the corresponding homomorphisms of algebras of sections of the structure sheafs are Poisson.

Note that a Poisson subvariety is naturally equipped with a structure of a Poisson variety. Clearly, open subvariety of a Poisson variety is Poisson.

Note that for any multiplicatively closed subset S of a Poisson algebra A the quotient algebra A_S is equipped with a unique Poisson bracket such that the natural homomorphism $A \rightarrow A_S$ is Poisson. Thus a Poisson bracket on $\mathbb{K}[X]$ defines the Poisson structure on X provided X is quasiprojective.

Proposition 2.1.3. *Let X be a Poisson variety. Then*

- (1) *Any irreducible component of X is a Poisson subvariety.*
- (2) *Suppose X is irreducible. The normalization \tilde{X} of X is equipped with a unique Poisson structure such that the canonical morphism $\pi : \tilde{X} \rightarrow X$ is Poisson.*

Proof. We may assume that X is affine. Recall that $\{f, \cdot\}$ is a derivation of $\mathbb{K}[X]$ for any $f \in \mathbb{K}[X]$. Assertion 2 was proved by Kaledin in [Ka]. Assertion 1 stems from the following lemma. \square

Lemma 2.1.4. *Suppose A is a Noetherian \mathbb{K} -algebra. Minimal prime ideals of A are stable under any derivation $D \in \text{Der}(A, A)$.*

Proof. Localizing at a minimal prime ideal, we may assume that A is a local Artinian ring with the maximal ideal \mathfrak{m} . Let $x \in \mathfrak{m}$. Choose an integer n such that $x^{n-1} \neq x^n = 0$. It remains to note that $0 = D(x^n) = nx^{n-1}Dx$. \square

2.2. Poisson bivector. At first, we recall the relation between bivectors and 2-forms on vector spaces. This material is standard, but we want to specify the choice of signs.

Let V be a finite dimensional vector space, $P \in \bigwedge^2 V$. The bivector P induces the linear map $v : V^* \rightarrow V$ by formula

$$(2.4) \quad \langle \alpha, v(\beta) \rangle = \langle P, \alpha \wedge \beta \rangle, \alpha, \beta \in V^*.$$

If P is nondegenerate, then v is an isomorphism. In general, P lies in $\bigwedge^2 v(V^*)$ and is a nondegenerate bivector in this space. The map $v : V^* \rightarrow V$ is the composition of the canonical surjection $V^* \rightarrow v(V^*)^*$, the isomorphism $v(V^*)^* \rightarrow v(V^*)$ induced by $P \in \bigwedge^2 v(V^*)$ and the embedding $v(V^*) \hookrightarrow V$.

We can define the skew-symmetric nondegenerate bilinear form ω_P on $v(V^*)$ by formula

$$(2.5) \quad \omega_P(v(\alpha), v(\beta)) = \langle P, \alpha \wedge \beta \rangle = \langle \alpha, v(\beta) \rangle$$

Now let ω be a nondegenerate skew-symmetric bilinear form on $U \subset V$. We may consider ω as a nondegenerate bivector in $\bigwedge^2 U^*$ and construct the bivector $P_\omega \in \bigwedge^2 U$ by formula (2.5). By embedding $\bigwedge^2 U$ into $\bigwedge^2 V$, we obtain the bivector in $\bigwedge^2 V$ with $v(V^*) = U$. The maps $P \mapsto \omega_P$, $\omega \mapsto P_\omega$ are inverse to each other.

Now let X be a variety. By a bracket on X we mean a skew-symmetric bilinear operation on O_X satisfying the Leibnitz identity. Brackets on X are in one-to-one correspondence with global bivectors, that is, global sections of the sheaf $\text{Hom}_{O_X}(\bigwedge^2 \Omega_X, O_X)$, where Ω_X is the sheaf of Kähler differentials on X . If X is smooth, then we get a bivector in the usual sense. If a bracket satisfies (2.2), then the corresponding bivector is called *Poisson*. The bracket corresponding to a bivector P is given by

$$(2.6) \quad \{f, g\} = P(df \wedge dg), f, g \in \mathbb{K}(X).$$

Now let P be a bivector on X and $x \in X$. Using the bivector P_x , we construct the linear map $v_x : T_x^* X \rightarrow T_x X$ defined by (2.4). Let f be a rational function on X . The vectors $v_x(df)$ form a vector field defined in the points of the definition of f . This vector field is called the *skew-gradient* of f , we denote it by $v(f)$. If P is a Poisson bivector, then, by the Jacobi identity for the bracket, the equality $L_{v(f)} P = 0$ holds, where L denotes the Lie derivative.

Put

$$(2.7) \quad T_x^P X = \text{im } v_x.$$

Clearly, $P_x \in \bigwedge^2 T_x^P X$. The bivector P_x induces the bilinear skew-symmetric nondegenerate form ω_x on $T_x^P X$ by formula (2.5).

Let $x \in X^{\text{reg}}$. On the open subvariety $X^{\text{max}} \subset X^{\text{reg}}$ consisting of all points x such that $\text{rk } P_x = \max_{y \in X^{\text{reg}}} \text{rk } P_y$ the spaces $T_x^P X$ form a locally trivial vector bundle denoted by $T^P X$. We have the global section ω of $\bigwedge^2 T^{P*} X$ over X^{max} equal to ω_x in x . In the sequel we often call this section a 2-form. If $X = X^{\text{max}}$, then P is said to have constant rank.

Definition 2.2.1. Let X be a smooth variety and V a locally-trivial (in étale topology) subbundle of TX . The subbundle V is called a *distribution* on X . The distribution V is called *involutive*, if for any sections ξ, η of V on any étale neighborhood of X the commutator $[\xi, \eta]$ is also a section of V .

The following proposition is standard (compare with [CdSW], Theorem 4.3, [AG], Subsection 3.2).

Proposition 2.2.2. *Let X be a smooth variety and P a bivector of constant rank on X . Then the following conditions are equivalent*

- (1) P is Poisson.
- (2) The distribution $T^P X$ is involutive and

$$(2.8) \quad \begin{aligned} &\omega([\xi, \eta], \zeta) + \omega([\eta, \zeta], \xi) + \omega([\zeta, \xi], \eta) = \\ &L_\xi(\omega(\eta, \zeta)) + L_\eta(\omega(\zeta, \xi)) + L_\zeta(\omega(\xi, \eta)). \end{aligned}$$

for all rational sections ξ, η, ζ of $T^P X$.

Note that P is uniquely determined by $T^P X$ and ω .

Definition 2.2.3. A Poisson variety X is called *symplectic*, if it is smooth and for any x from any irreducible component $X_0 \subset X$ the equality $\text{rk}_x P = \dim X_0$ holds.

Definition 2.2.4. An irreducible Poisson variety X is said to be *generically symplectic*, if X^{max} is symplectic.

If a Poisson variety X is symplectic (resp., generically symplectic), then ω is a symplectic form in the usual sense on X (resp., on X^{max}).

2.3. Examples of Poisson varieties.

Example 2.3.1. Let \mathfrak{g} be an algebraic Lie algebra. The algebra $\mathbb{K}[\mathfrak{g}^*] \cong S(\mathfrak{g})$ possesses a unique Poisson bracket $\{\cdot, \cdot\}$ such that $\{\xi, \eta\} = [\xi, \eta]$ for all $\xi, \eta \in \mathfrak{g}$. Thus \mathfrak{g}^* is equipped with the structure of a Poisson variety. The corresponding Poisson bivector P is given by $P_\alpha(\xi \wedge \eta) = \langle \alpha, [\xi, \eta] \rangle$, $\alpha \in \mathfrak{g}^*$. This implies $T_x^P = \mathfrak{g}_*x$. A locally-closed subvariety $X \subset \mathfrak{g}^*$ is Poisson iff X is $\text{Int}(\mathfrak{g})$ -stable. In particular, an orbit O of the action $\text{Int}(\mathfrak{g}) : \mathfrak{g}^*$ is a Poisson subvariety in \mathfrak{g}^* . This variety is symplectic, the corresponding symplectic form ω is called *the Kostant-Kirillov form*. Explicitly, $\omega_\alpha(\xi_*, \eta_*) = \alpha([\xi, \eta])$, $\alpha \in O$, $\xi, \eta \in \mathfrak{g}$.

Example 2.3.2. Let Y be a variety and $\mathcal{V}ect$ its sheaf of vector fields. The vector bundle $T^*Y = \text{Spec}(S_{O_Y}(\mathcal{V}ect))$ is called the *cotangent bundle* of Y . The cotangent bundle is locally trivial iff Y is smooth. Let us equip $X = T^*Y$ with a natural Poisson structure. It is enough to do it locally and check that the obtained structures are compatible. Thus one may assume that $Y = \text{Spec}(A)$ for an algebra A of finite type. In this case $\mathbb{K}[X] = S_A(D)$, where $D = \text{Der}(A, A)$ is the module of derivations of A . The algebra $S_A(D)$ possesses a unique Poisson bracket $\{\cdot, \cdot\}$ such that

$$\begin{aligned} \{f_1, f_2\} &= 0, \{f_1, d_1\} = d_1(f_1), \{d_1, d_2\} = [d_1, d_2] := d_2d_1 - d_1d_2, \\ f_1, f_2 &\in A, d_1, d_2 \in D. \end{aligned}$$

The uniqueness follows from the fact that A and D generate $S_A(D)$. Let us sketch the proof of the existence. Firstly, using the construction of a tensor algebra, we prove that any $x \in D, a \in A$ define derivations of the algebra $T_A(D)$ (commutators with these elements). Then, considering $S_A(D)$ as a quotient of $T_A(D)$, one can prove that a, x define the derivations d_a, d_x of $S_A(D)$. Using an analogous argument and the Leibnitz rule, we can construct the derivation d_f of $S_A(D)$ corresponding to $f \in S_A(D)$. The bracket $\{g, f\} = d_f(g)$ has the required properties.

The compatibility of these brackets follows from the uniqueness property.

If Y is smooth, the Poisson structure constructed above coincides with the standard symplectic structure on T^*Y (see. [V3], Ch. 2, Section 1.4).

Note that, by construction, any regular vector field on Y defines an element in $\mathbb{K}[T^*Y]$.

Example 2.3.3. Let X, Y be Poisson varieties. The product $X \times Y$ is naturally equipped with a Poisson structure.

Example 2.3.4. Let X be a Poisson variety, Y a variety, $\varphi : Y \rightarrow X$ an étale morphism. Let us show that Y possesses a unique Poisson structure such that φ is a Poisson morphism. Let P_X be the Poisson bivector on X . There is a unique bivector P_Y on Y such that $d\varphi P_Y = P_X$. It remains to show that P_Y is a Poisson bivector. The latter is an easy consequence of Proposition 2.2.2.

Example 2.3.5. Let X be a Poisson variety, Y a normal irreducible variety, $\varphi : Y \rightarrow X$ a morphism. Suppose that φ is étale in any point of an open subset $Y^0 \subset Y^{reg}$ such that $\text{codim}_Y(Y \setminus Y^0) \geq 2$. Let us show that Y possesses a unique Poisson structure such that φ is a Poisson morphism. By the previous example, Y^0 possesses a unique Poisson structure

such that $\varphi|_{Y^0}$ is a Poisson morphism. Since Y is normal, $\mathbb{K}[U] = \mathbb{K}[U \cap Y^0]$ for any open subset $U \subset Y$. This allows one to define a Poisson structure on the whole variety Y . Clearly, φ is a Poisson morphism.

2.4. Stratification of a Poisson variety. The following proposition appeared in [Pol], Section 2. The proof is essentially contained in Corollaries 2.3, 2.4.

Proposition 2.4.1. *Let X be a Poisson variety. There exists a unique decomposition of X into the disjoint union of irreducible locally closed subvarieties X_i fulfilling the following conditions:*

- (a) $\overline{X_i}$ is a Poisson subvariety of X .
- (b) $X_i = \overline{X_i}^{\max}$.

3. HAMILTONIAN ACTIONS

In Subsection 3.1 we define Hamiltonian actions of reductive groups on Poisson varieties and study their simplest properties. In Subsections 3.2, 3.3 we introduce some examples of Hamiltonian (respectively, conical Hamiltonian) varieties. In Subsection 3.4 we describe some classes of equidefectinal and strongly equidefectinal actions. Finally, in Subsection 3.5 we use Hamiltonian actions of tori to prove the Zariski-Nagata theorem on the purity of branch locus.

In this section X is a Poisson variety (not necessarily smooth or irreducible) and G is a reductive group acting on X by Poisson automorphisms.

3.1. Main definitions and some properties. Assume that there is a linear map $\xi \mapsto H_\xi$ from \mathfrak{g} to $\mathbb{K}[X]$ satisfying the following two conditions:

- (H1) $L_{\xi_*}f = \{H_\xi, f\}$ for any $f \in \mathbb{K}(X)$, $\xi \in \mathfrak{g}$.
- (H2) The map $\xi \mapsto H_\xi$ is G -equivariant.

Definition 3.1.1. An action $G : X$ together with a linear map $\xi \mapsto H_\xi$ satisfying (H1), (H2) is called *Hamiltonian*. If the action $G : X$ is Hamiltonian, then X is said to be a Hamiltonian G -variety. The functions H_ξ are called the *hamiltonians* of the action. The morphism $\mu_{G,X} : X \rightarrow \mathfrak{g}^*$ defined by $\langle \mu_{G,X}(x), \xi \rangle = H_\xi(x)$ for all $x \in X$, $\xi \in \mathfrak{g}$ is called the *moment map*.

Since $\{H_\xi, H_\eta\} = L_{\xi_*}H_\eta = H_{[\xi, \eta]}$, $\mu_{G,X}$ is a Poisson morphism.

When X is symplectic, our definition coincides with the standard one, see, for example, [V3], Ch.2, §2.

In the sequel we fix a G -invariant symmetric bilinear form $(\xi, \eta) = \text{tr}_V(\xi\eta)$ on \mathfrak{g} , where V is a locally effective G -module. This form is nondegenerate on any subalgebra $\mathfrak{h} \subset \mathfrak{g}$ corresponding to a reductive subgroup $H \subset G$. We identify \mathfrak{h}^* with \mathfrak{h} using this form. So one may consider the moment map as a morphism $X \rightarrow \mathfrak{g}$.

Let $\psi_{G,X} : X \rightarrow \mathfrak{g}/G$ be the morphism defined by $\psi_{G,X} = \pi_{G,\mathfrak{g}} \circ \mu_{G,X}$.

Remark 3.1.2. If H is a normal subgroup in G and X is a Hamiltonian G/H -variety, then X is naturally endowed with the structure of a Hamiltonian G -variety. The moment map $\mu_{G,X}$ is the composition of $\mu_{G/H,X}$ and the natural embedding $(\mathfrak{g}/\mathfrak{h})^* \hookrightarrow \mathfrak{g}^*$. Conversely, if X is a Hamiltonian G -variety and a normal subgroup $H \subset G$ acts trivially on X , we can consider X as a Hamiltonian G/H -variety with $\mu_{G/H,X} = \pi \circ \mu_{G,X}$, where $\pi : \mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{h}$ is a canonical projection. Note that any fiber of $\psi_{G,X}$ is contained in a fiber of $\psi_{G/H,X}$.

Definition 3.1.3. Let X_1, X_2 be Hamiltonian G -varieties, $\varphi : X_1 \rightarrow X_2$ a G -equivariant Poisson morphism. The morphism φ is said to be *Hamiltonian*, if $\mu_{G,X_1} = \mu_{G,X_2} \circ \varphi$. If Y is a G -stable subvariety of X such that the embedding $Y \hookrightarrow X$ is Hamiltonian, then Y is said to be a Hamiltonian subvariety of X .

Proposition 3.1.4. *Let X be a Hamiltonian G -variety and $x \in X^{\max}$. For $v \in T_x^P X, \xi \in \mathfrak{g}$, we have*

$$\langle d_x \mu_{G,X}(v), \xi \rangle = \omega_x(\xi_* x, v).$$

In particular, $d_x \mu_{G,X}(T_x^P X) = \mathfrak{g}_x^\perp$.

Proof. The proof is completely analogous to the symplectic case, considered, for example, in [V3], Chapter 2, Subsection 2.4. \square

We recall that the rank and the upper and lower defects of a G -irreducible Hamiltonian G -variety were defined in Definition 1.2.2.

Remark 3.1.5. If X is symplectic, our definition of the defect coincides with that given in [V3], Subsection 2.5.

The following properties of the rank and the defects follow directly from the definition.

Lemma 3.1.6. (1) *Suppose that X is a G -irreducible Hamiltonian G -variety and X_0 is an irreducible component of X . Then $\underline{\text{def}}_G(X) = \underline{\text{def}}_{G^\circ}(X_0)$, $\overline{\text{def}}_G(X) = \overline{\text{def}}_{G^\circ}(X_0)$, $\text{rk}_G(X) = \text{rk}_{G^\circ}(X_0)$.*
 (2) *Let X_1, X_2 be G -irreducible Hamiltonian G -varieties, $\varphi : X_1 \rightarrow X_2$ a dominant generically finite Hamiltonian morphism. Then $\underline{\text{def}}_G(X_1) = \underline{\text{def}}_G(X_2)$, $\overline{\text{def}}_G(X_1) = \overline{\text{def}}_G(X_2)$, $\text{rk}_G(X_1) = \text{rk}_G(X_2)$.*

The next proposition is the main property of the upper and lower defects.

Proposition 3.1.7. *Let X be G -irreducible. There are the inequalities*

$$\underline{\text{def}}_G(X) \leq \overline{\text{def}}_G(X), m_G(X) \leq \dim \overline{\mu_{G,X}(X)},$$

simultaneously turning into equalities. If X is generically symplectic, then these inequalities turn into equalities.

Proof. The case when X is symplectic can be found in [V3], Chapter 2, Subsections 2.4, 2.6. In the general case the proof is analogous. \square

Remark 3.1.8. Let us give an example when the inequalities of Proposition 3.1.7 are strict. Let X be an irreducible Poisson variety with the zero Poisson bracket. Let the torus $(\mathbb{K}^\times)^n$ act trivially on X . For the moment map one may take an arbitrary nonconstant morphism $X \rightarrow \mathbb{K}^n$.

Corollary 3.1.9. *The following assertions are equivalent:*

- (1) $\underline{\text{def}}_G(X) = \overline{\text{def}}_G(X) = \text{rk } G$.
- (2) $m_G(X) = \dim G$.

Proof. (1) \Rightarrow (2). By Proposition 3.1.7, $\dim \overline{\mu_{G,X}(X)} // G = \text{rk } G$. Since a general fiber of $\pi_{G,\mathfrak{g}}$ is a single orbit, it follows that $\overline{\mu_{G,X}(X)} = \mathfrak{g}$. But $\underline{\text{def}}_G(X) = \overline{\text{def}}_G(X)$. Proposition 3.1.7 implies $m_G(X) = \dim G$.

The proof of (2) \Rightarrow (1) is analogous. \square

3.2. Examples of Hamiltonian varieties.

Example 3.2.1. Let X be a locally closed G -stable subvariety in $\mathfrak{g} \cong \mathfrak{g}^*$. By Example 2.3.1, X is a Poisson variety. Put $H_\xi = \xi|_X \in \mathbb{K}[X]$. It is checked directly that the pair $X, (H_\xi)$ satisfies the conditions (H1),(H2). So X is a Hamiltonian G -variety. The moment map is the embedding $X \hookrightarrow \mathfrak{g}$.

Example 3.2.2. Let Y be a G -variety and $X = T^*Y$ (see Example 2.3.2). Being a global vector field on Y , the velocity vector field $\xi_*, \xi \in \mathfrak{g}$, defines the function $H_\xi \in \mathbb{K}[X]$. The group G acts naturally on X . The pair $X, (H_\xi)$ clearly satisfies (H2). Note that $L_{\xi_*}\eta = [\xi_*, \eta]$ for any open subset $U \subset Y, \eta \in \mathcal{V}ect(U), \xi \in \mathfrak{g}$. It follows from the construction of the Poisson structure that the pair $X, (H_\xi)$ satisfies (H1). The moment map is given by $\langle \mu_{G,X}(y, \alpha), \xi \rangle = \langle \alpha, \xi_*y \rangle, y \in Y, \alpha \in T_y^*Y, \xi \in \mathfrak{g}$.

If Y is smooth, the Hamiltonian structure constructed above coincides with the standard one, see [V3], Ch.2, §2, Example 1.

Example 3.2.3. Let X be a Hamiltonian G -variety, \tilde{X} its normalization, $\varphi : \tilde{X} \rightarrow X$ the canonical morphism. The G -variety \tilde{X} can be equipped with a unique Hamiltonian structure such that φ is a Hamiltonian morphism.

Example 3.2.4. Let X be a Hamiltonian G -variety, Y a Poisson subvariety of X . Since the ideal sheaf of Y is stable under the brackets with $H_\xi, \xi \in \mathfrak{g}$, the subvariety $Y \subset X$ is G° -stable. If $GY = Y$, then the pair $(Y, (H_\xi|_Y))$ satisfies (H1),(H2). Thus Y becomes a Hamiltonian subvariety of X .

Example 3.2.5. Suppose X_1, X_2 are Poisson varieties, groups G_1, G_2 act on X_1, X_2 , respectively, and these actions are Hamiltonian. Then the action $G_1 \times G_2 : X_1 \times X_2$ is Hamiltonian. The moment map is given by the formula $\mu_{G_1 \times G_2, X_1 \times X_2}(x_1, x_2) = \mu_{G_1, X_1}(x_1) + \mu_{G_2, X_2}(x_2)$ for $x_1 \in X_1, x_2 \in X_2$.

Example 3.2.6. Let X be a Hamiltonian G -variety, Y an irreducible normal G -variety and $\varphi : Y \rightarrow X$ a G -equivariant morphism satisfying the assumptions of Example 2.3.5. Then Y is equipped with a unique Poisson structure such that φ is a Poisson morphism. Since this structure is unique, it is G -invariant. Let H_ξ^X be the hamiltonians for the action $G : X$. Put $H_\xi^Y = \varphi^*(H_\xi^X)$. Clearly, the pair $Y, (H_\xi^Y)$ satisfies (H2). It is easily deduced from the uniqueness of a lifting of a derivation ([Le], Chapter 10) that this pair satisfies also (H1). Note that $\mu_{G,Y} = \mu_{G,X} \circ \varphi$ whence φ is a Hamiltonian morphism.

Example 3.2.7. In particular, let $\eta \in \mathfrak{g}$, H be a subgroup of finite index in G_η . Since all adjoint orbits have even dimension, the algebra $\mathbb{K}[G\eta]$ is finitely generated (see. [PV2], Section 3.7). But $\mathbb{K}[G/H]$ is the integral closure of $\mathbb{K}[G\eta]$ in $\mathbb{K}(G/H)$. Thus $\mathbb{K}[G/H]$ is finitely generated. The natural morphism $\varphi : \text{Spec}(\mathbb{K}[G/H]) \rightarrow \overline{G\eta}$ satisfies the assumptions of Example 3.2.6. So $X = \text{Spec}(\mathbb{K}[G/H])$ is equipped with the structure of a Hamiltonian G -variety. The equality $\mu_{G,X} = \varphi$ holds. Note that G/H is an open subset in $\text{Spec}(\mathbb{K}[G/H])$ and thus a Hamiltonian G -variety. This variety is symplectic.

Example 3.2.8. There is an important special case of the previous construction. Let $G = \text{Sp}(V)$, where V is a symplectic vector space with a constant symplectic form ω , and η a highest weight vector of \mathfrak{g} . Then the G -variety V coincides with $\text{Spec}(\mathbb{K}[G/(G_\eta)^\circ])$. The corresponding Poisson bivector corresponds to ω . The moment map is given by (see [V3], Chapter 2, Example 2) $\langle \mu_{G,V}(v), \xi \rangle = \frac{1}{2}\omega(\xi v, v), v \in V, \xi \in \mathfrak{g}$.

Example 3.2.9. Let X be a Hamiltonian G -variety with the hamiltonians $H_\xi, \xi \in \mathfrak{g}$, and H a reductive subgroup in G . Then the H -variety X and the linear map $\mathfrak{h} \rightarrow \mathbb{K}[X], \xi \mapsto H_\xi$, satisfy (H1),(H2). Thus $H : X$ is a Hamiltonian action. The moment map $\mu_{H,X}$ is the composition of $\mu_{G,X}$ and the restriction map $\mathfrak{g}^* \rightarrow \mathfrak{h}^*$. In particular, any linear action of a reductive group on a symplectic vector space becomes Hamiltonian.

Example 3.2.10. Suppose X is a Hamiltonian G -variety, and a reductive group H acts on X by Hamiltonian automorphisms. Suppose the good categorical quotient $X//H$ exists (for example, X is affine or X is quasiprojective and H is finite). Then $X//H$ is equipped with a unique structure of a Hamiltonian G -variety such that $\pi_{H,X}$ is a Hamiltonian morphism.

3.3. Conical Hamiltonian varieties. A conical Hamiltonian variety was defined in the Introduction, Definition 1.2.6.

Example 3.3.1. Let H be a reductive group. The group \mathbb{K}^\times acts on \mathfrak{h} as usual, that is $(t, x) \mapsto tx, t \in \mathbb{K}^\times, x \in \mathfrak{h}$. Let X be a closed H -stable cone in \mathfrak{h} (a cone in \mathfrak{h} , is, by definition, a \mathbb{K}^\times -stable subset). For any reductive subgroup $G \subset H$ the Hamiltonian G -variety X (see Examples 3.2.7 and 3.2.9) equipped with the action of \mathbb{K}^\times induced from \mathfrak{h} is conical of degree 1.

Example 3.3.2. Let $G : V$ be a linear Hamiltonian action (see Examples 3.2.8, 3.2.9). The Hamiltonian G -variety V together with the action $\mathbb{K}^\times : V$ given by $(t, v) \mapsto tv$ is conical of degree 2.

Example 3.3.3. Let Y be an affine G -variety and $X = T^*Y$ (see Example 3.2.2). The variety X is a vector bundle over Y . Therefore there is the action $\mathbb{K}^\times : X$ by the fiberwise multiplication. The Hamiltonian G -variety X equipped with this action of \mathbb{K}^\times is conical of degree 1.

Example 3.3.4. If X is a conical Hamiltonian G -variety, then any G -stable union of irreducible components of X is a conical variety.

Example 3.3.5. Let X be a conical Hamiltonian G -variety. The action $\mathbb{K}^\times : X$ can be lifted to the action of \mathbb{K}^\times on the normalization \tilde{X} of X . The Hamiltonian G -variety \tilde{X} equipped with this action is conical.

The following lemma describes some basic properties of conical varieties.

Lemma 3.3.6. *Let X be a conical Hamiltonian G -variety of degree k . Then*

- (1) $0 \in \text{im } \psi_{G,X}$.
- (2) *Suppose X is irreducible and normal. Then the subalgebra $\mathbb{K}[C_{G,X}] \subset \mathbb{K}[X]^G$ (see the Introduction for the definition of $C_{G,X}$) is \mathbb{K}^\times -stable. The morphisms $\tilde{\psi}_{G,X} : X \rightarrow C_{G,X}, \tau_{G,X} : C_{G,X} \rightarrow \mathfrak{g}//G$ are \mathbb{K}^\times -equivariant, where the action $\mathbb{K}^\times : \mathfrak{g}//G$ is induced from the action $\mathbb{K}^\times : \mathfrak{g}$ given by $(t, x) \mapsto t^k x, t \in \mathbb{K}^\times, x \in \mathfrak{g}$.*
- (3) *Under the assumptions of the previous assertion there exists a unique point $\lambda_0 \in C_{G,X}$ such that $\tau_{G,X}(\lambda_0) = 0$. For any point $\lambda \in C_{G,X}$ the limit $\lim_{t \rightarrow 0} t\lambda$ exists and is equal to λ_0 .*

Proof. Let $x \in X$. It follows from (Con1) that there exists the limit $y = \lim_{t \rightarrow 0} t\pi_{G,X}(x)$. By (Con2), $\mu_{G,X} // G(y) = 0$. This proves the first assertion.

The morphism $\psi_{G,X} : X \rightarrow \mathfrak{g}//G$ is \mathbb{K}^\times -equivariant by (Con2). Therefore the integral closure of $\psi_{G,X}^*(\mathbb{K}[\mathfrak{g}]^G)$ in $\mathbb{K}(X)^G$, that is, $\mathbb{K}[C_{G,X}]$, is \mathbb{K}^\times -stable. This proves assertion 2.

Assertion 3 follows easily from the observations that $C_{G,X}$ is irreducible and $\tau_{G,X}$ is \mathbb{K}^\times -equivariant. \square

3.4. Equidefectinal and strongly equidefectinal Hamiltonian varieties. The definitions of equidefectinal and strongly equidefectinal Hamiltonian varieties were given in the Introduction (Definitions 1.2.4, 1.2.8).

Lemma 3.4.1. *Suppose X is a G -irreducible Hamiltonian G -variety such that $m_G(X) = \dim G$ or some component of X is generically symplectic. Then X is equidefectinal.*

Proof. If $m_G(X) = \dim G$ this follows directly from Proposition 3.1.7 and Corollary 3.1.9. Now suppose X is generically symplectic. By Proposition 3.1.7, $m_G(X) = \dim \overline{\text{im } \mu_{G,X}}$. Note that $\overline{\text{im } \psi_{G,X}} = \overline{\text{im } \mu_{G,X}} // G$. Every fiber of the quotient morphism $\overline{\text{im } \mu_{G,X}} \rightarrow \overline{\text{im } \psi_{G,X}}$ consists of finitely many orbits. Therefore the maximal dimension of a fiber coincides with $\text{rk}_G(X)$. Hence $m_G(X) - \text{rk}_G(X) = \dim \overline{\text{im } \psi_{G,X}}$. \square

Lemma 3.4.2. *Suppose that X is a Hamiltonian G -variety such that any stratum described in Proposition 2.4.1 is a symplectic variety. Then X is strongly equidefectinal. In particular, any symplectic variety X is strongly equidefectinal.*

Proof. This follows easily from Lemma 3.4.1. \square

Lemma 3.4.3. *Let X, Y be Poisson varieties and $\varphi : X \rightarrow Y$ a Poisson morphism. Then*

- (1) *For any Poisson subvariety $X' \subset X$ the closure of $\varphi(X')$ is a Poisson subvariety of Y .*
- (2) *Suppose that the stratification of Y consists of symplectic varieties. Then the same is true for X provided φ is finite and dominant.*

Proof. The first assertion is straightforward.

Proceed to assertion 2. Note that any irreducible Poisson subvariety of Y is generically symplectic. Let X' be an irreducible locally closed Poisson subvariety of X . The subvariety $\overline{\varphi(X')} \subset Y$ is Poisson. Since φ is finite, the ranks of Poisson bivectors on X_1 and on $\varphi(X_1)$ coincide. Thus X' is generically symplectic. It remains to apply this observation to the strata of X . \square

Corollary 3.4.4. *Let G be a connected reductive group, G_0 its reductive subgroup, $\eta \in \mathfrak{g}$ and H a subgroup of finite index in G_η . The Hamiltonian G_0 -variety $\text{Spec}(\mathbb{K}[G/H])$ is strongly equidefectinal.*

Proof. Apply Lemma 3.4.3 to the morphism $\text{Spec}(\mathbb{K}[G/H]) \rightarrow \overline{G\eta}$ and use Lemma 3.4.2. \square

The following lemma is used in the proof of Theorem 1.2.9.

Lemma 3.4.5. *Let X, Y be Hamiltonian G -varieties and $\varphi : X \rightarrow Y$ an étale Hamiltonian morphism. If Y is strongly equidefectinal, then so is X .*

Proof. Let $Y = \coprod_i Y_i$ be the stratification of Y satisfying the claims of Definition 1.2.8. Since φ is étale, one can see that $X_i = \varphi^{-1}(Y_i)$ is a Poisson subvariety of X . The morphism $\varphi|_{X_i} : X_i \rightarrow Y_i$ is automatically Poisson and étale. Thence X_i is smooth. Let $X_i = \coprod X_i^j$ be a unique stratification of X_i by G -irreducible unions of components. This stratification satisfies the requirements of Definition 1.2.8. \square

3.5. An application of Hamiltonian actions: the Zariski-Nagata theorem on the purity of branch locus. In this subsection we generalize the Zariski-Nagata theorem, see, for example, [D], Chapter 4, Subsection 1.4. However, our generalization can be easily deduced from this theorem.

Proposition 3.5.1. *Let X, Y be irreducible varieties of the same dimension, $\varphi : Y \rightarrow X$ a dominant morphism. Suppose Y is normal, X is smooth. Then the complement in Y to the open subset $Y^0 = \{y \in Y \mid \varphi \text{ is étale in } y\}$ is a subvariety of pure codimension 1.*

Proof. Removing all components of $Y \setminus Y^0$ of codimension 1, we may assume that $\text{codim}_Y Y \setminus Y^0 \geq 2$. Further, any point $x \in X$ has a neighborhood admitting an étale morphism to \mathbb{A}^n (see, for example, [D], Chapter 2, Subsection 6.3.). So we may assume $X = \mathbb{A}^n$, Y is affine. Put $T = (\mathbb{K}^\times)^n$. The action $T : T^*T \cong T \times \mathbb{A}^n$ is Hamiltonian (Example 3.2.2). Consider the action $T : T \times Y$, where T acts on Y trivially and on T by left translations, and the T -equivariant morphism $\Phi : T \times Y \rightarrow T \times \mathbb{A}^n$, $\Phi(t, y) = (t, \varphi(y))$. This morphism satisfies the conditions of Example 3.2.6, thus the action $T : T \times Y$ is Hamiltonian. If Y is smooth, then φ is étale because $Y \setminus Y^0$ is the set of zeroes of the Jacobian of φ . Assume that Y is not smooth. Corollary 2.4 in [Pol] implies that $T \times Y^{\text{sing}} = (T \times Y)^{\text{sing}}$ is a Poisson subvariety in $T \times Y$. The action $T : (T \times Y)^{\text{sing}} = T \times Y^{\text{sing}}$ is Hamiltonian (Example 3.2.4). Choose an irreducible component $Z \subset T \times Y^{\text{sing}}$. We see that $\overline{\text{def}}_T(Z) \leq \dim Z // T = \dim Y^{\text{sing}} < \dim Y = \dim T = \underline{\text{def}}_T(Z)$. This contradicts Proposition 3.1.7. \square

4. CENTRAL-NILPOTENT HAMILTONIAN VARIETIES

Throughout this section G is a connected reductive algebraic group, X is an irreducible quasiprojective CN Hamiltonian G -variety (see Definition 1.3.1).

Since X is CN as a Hamiltonian G -variety, X is CN also as a (G, G) -variety. In other words, the image of $\mu_{(G, G), X}$ consists of nilpotent elements. The number of nilpotent orbits is finite. Thus there is a unique open orbit $G\eta \subset \overline{\text{im } \mu_{(G, G), X}}$. Put $X^0 = \mu_{G, X}^{-1}(G\eta)$. This is a G -stable open subset of X .

In the first two subsections we study the structure of the Hamiltonian G -variety X . In Subsection 4.1 we describe the variety X^0 for an arbitrary normal variety X . In Subsection 4.2 we describe the whole variety X provided that X is, in addition, affine. The description is carried out for groups G such that $Z(G)^\circ \cap (G, G) = \{1\}$, in this case $G \cong Z(G)^\circ \times (G, G)$. This requirement is not restrictive: any connected reductive group possesses a covering satisfying the requirement.

In Subsection 4.3 we prove that the dimension of a fiber of $\psi_{G, X}$ for an irreducible CN Hamiltonian variety X does not exceed $\dim X - \underline{\text{def}}_G(X)$. This gives us a proof of Theorem 1.2.3 for CN varieties.

In Subsections 4.4 and 4.5 Theorems 1.2.5, 1.2.9 are proved for central-nilpotent varieties X .

4.1. The structure of X^0 . Let X be normal, $G \cong Z(G)^\circ \times (G, G)$, η, X^0 such as above. We put $O := G/(G_\eta)^\circ \cong (G, G)/((G, G)_\eta)^\circ$. This is a Hamiltonian G -variety (see Example 3.2.7).

Let X_0 be a quasiprojective Hamiltonian $Z(G)^\circ$ -variety. Since $G = Z(G)^\circ \times (G, G)$, we can consider X_0 as a Hamiltonian G -variety. Let Γ be a finite group acting on $X_0 \times O$ by Hamiltonian automorphisms. The quotient $(X_0 \times O)/\Gamma$ is equipped with the natural Hamiltonian structure (see Example 3.2.10).

The main result of this subsection is the following

Theorem 4.1.1. *There exist a Hamiltonian $Z(G)^\circ$ -variety X_0 and a finite group Γ acting freely on $X_0 \times O$ by Hamiltonian automorphisms such that $X^0 \cong (X_0 \times O)/\Gamma$ (the isomorphism of Hamiltonian G -varieties).*

The theorem is proved in the following lemmas.

Lemma 4.1.2. *$((G, G)_{\mu_{G,X}(x)})^\circ \subset (G, G)_x \subset (G, G)_{\mu_{G,X}(x)}$ for any $x \in X^0$.*

Proof. By the choice of η , $\overline{\text{im } \mu_{(G,G),X}} = \overline{G\eta}$. Proposition 3.1.7 implies that $\dim(G, G)x = \dim(G, G)\eta$. Since $(G, G)_x \subset (G, G)_{\mu_{G,X}(x)}$, we are done. \square

Lemma 4.1.3. *Let G be an algebraic group, H its subgroup, Y a quasiprojective H -scheme, $X = G *_H Y$ (this homogeneous bundle exists by [PV2], Section 4.8). Then*

- (1) *Y is H -irreducible iff X is G -irreducible.*
- (2) *X is normal iff Y is so.*

Proof. The first assertion follows directly from the definition of homogeneous bundles.

Proceed to assertion 2. Clearly, if Y is a normal variety, then so is $G *_H Y$. Suppose now that $G *_H Y$ is a normal variety. Let Y_{red} denote the variety associated with Y . The canonical morphism $G *_H Y \rightarrow G *_H Y_{red}$ is an isomorphism. Hence $Y = Y_{red}$. If \tilde{Y} is the normalization of Y , then $G *_N \tilde{Y}$ is the normalization of $G *_N Y$. Thus Y is normal. \square

Lemma 4.1.4. *The morphism of schemes $\varphi : G *__{G_\eta} \mu_{(G,G),X}^{-1}(\eta) \rightarrow X$ is an open embedding with the image X^0 . The subscheme $\mu_{(G,G),X}^{-1}(\eta) \subset X$ is a normal G_η -irreducible subvariety.*

Proof. Obviously $\text{im } \varphi = X^0$. On the other hand, the morphism $\mu_{(G,G),X^0} : X^0 \rightarrow G/G_\eta$ is G -equivariant. So φ induces an isomorphism of G -schemes $X^0 \cong G *__{G_\eta} \mu_{(G,G),X}^{-1}(\eta)$. By Lemma 4.1.3, $\mu_{(G,G),X}^{-1}(\eta)$ is normal and G_η -irreducible. \square

Choose a component X_0 of $\mu_{(G,G),X}^{-1}(\eta)$ and denote by H its stabilizer in the group G_η . Note that $(G_\eta)^\circ \subset H$ and that $X^0 \cong G *_H X_0$. The action of $((G, G)_\eta)^\circ$ on X_0 is trivial by Lemma 4.1.2.

Put $\Gamma = H/((G_\eta)^\circ) \cong (H \cap (G, G))/((G, G)_\eta)^\circ$. This is a finite group acting freely on O by Hamiltonian automorphisms (the action is by the right translations). Since the action of $((G, G)_\eta)^\circ$ on X_0 is trivial, Γ acts also on X_0 by $Z(G)^\circ$ -automorphisms. Further, $\Gamma \cong H/H^\circ$ acts on $G *_H X_0$, $\gamma[g, x] = [g\tilde{\gamma}^{-1}, \tilde{\gamma}x]$, $g \in G, x \in X_0, \gamma \in \Gamma$, where $\tilde{\gamma}$ is an element from H mapping to γ under the natural projection $H \rightarrow H/H^\circ$. So the natural morphism $G *_H X_0 \rightarrow G *_H X_0 \cong X^0$ is the quotient for the action of Γ .

Consider the natural morphism $(G, G) \times X_0 \rightarrow G *_H X_0, (g, x) \mapsto [g, x]$. Since $((G, G)_\eta)^\circ$ acts trivially on X_0 , this morphism factors through $O \times X_0 \cong (G, G)/((G, G)_\eta)^\circ \times X_0 \rightarrow G *_H X_0$. The latter is clearly an isomorphism. The action of Γ on $O \times X_0$ as on the product of Γ -varieties coincides with the action on $G *_H X_0$. This action is free.

To complete the proof of Theorem 4.1.1 it remains to prove the following

Lemma 4.1.5. *There exists a Poisson bracket on X_0 such that the action $Z(G)^\circ : X_0$ is Hamiltonian with the moment map $\mu_{G,X}|_{X_0} - \eta$, the group Γ acts on X_0 by Hamiltonian automorphisms and the morphism $O \times X_0 \rightarrow X^0$ is Hamiltonian.*

Proof. The morphism $\pi_{\Gamma, O \times X_0} : O \times X_0 \rightarrow X^0$ is étale. Lift the Hamiltonian structure from X^0 to $G *_H X_0 \cong O \times X_0$. Clearly, Γ acts on $O \times X_0$ by Hamiltonian automorphisms.

Let us introduce a Poisson bracket on X_0 . For $x \in X_0$ the map $\mu_{(G,G),X}$ is a covering $(G,G)x \rightarrow G\eta$ (Lemma 4.1.2) thus $(G,G)x$ is equipped with the Poisson bracket lifted from $G\eta$. We denote this bracket by $\{\cdot, \cdot\}^{(G,G)x}$.

Identify X_0 with $X_0 \times \{eH^\circ\} \subset X_0 \times O$. For $f, g \in O_{O \times X_0, x}$ and $y \in X_0$ put

$$(4.1) \quad \{f, g\}^{X_0}(y) = \{f, g\}(y) - \{f|_{(G,G)y}, g|_{(G,G)y}\}^{(G,G)y}.$$

$\{f, g\}^{X_0}$ is an element of $O_{X_0, x}$. For $t \in Z(G)^\circ, y \in X_0$ the equality $\{tf, tg\}^{X_0}(ty) = \{f, g\}^{X_0}(y)$ holds. Let us check that $\{\cdot, \cdot\}^{X_0}$ is a Poisson bracket on $\mathbb{K}(X_0)$. In the proof we may assume that G is semisimple.

Denote by P the Poisson bivector of the variety $O \times X_0$. Let us show that $P_x \in \bigwedge^2 T_x X_0 \oplus \bigwedge^2 \mathfrak{g}_* x$ for $x \in X_0$ in general position. We may assume that X_0 is affine and smooth. We shall see now that for $f \in \mathbb{K}[X_0] \subset \mathbb{K}[X_0 \times O], g \in \mathbb{K}[O] \subset \mathbb{K}[X_0 \times O]$ the equality $\{f, g\} = 0$ holds. Note that $f \in \mathbb{K}[X_0 \times O]^G$ whence f commutes with any hamiltonian $H_\xi, \xi \in \mathfrak{g}$. Note that H_ξ is constant on X_0 . Since the space $\text{Span}_{\mathbb{K}}(H_\xi, \xi \in \mathfrak{g})$ is G -stable, $H_\xi \in \mathbb{K}[O]$. Moreover, $\mathbb{K}[O]$ is algebraic over the subalgebra generated by $H_\xi, \xi \in \mathfrak{g}$. It follows from the uniqueness property of a lifting of a derivation that $\{\mathbb{K}[X_0], \mathbb{K}[O]\} = 0$.

Since P is G -invariant, the projection of P_x to $\bigwedge^2 T_x X_0$ for $x \in X_0$ depends only on the X_0 -component of x . This projection is a Poisson bivector on X_0 . On the other hand, this is the bivector corresponding to the bracket $\{\cdot, \cdot\}^{X_0}$. So X_0 is Poisson, and the Poisson structure on $O \times X_0$ is the product structure.

Now let G be not necessarily semisimple. It remains to show that the action $Z(G)^\circ : X_0$ is Hamiltonian with the moment map $\mu := \mu_{G,X}|_{X_0} - \eta$. Clearly, μ is $Z(G)^\circ$ -invariant. Recall that G acts on $X_0 \times O$ as on the product of G -varieties and the action of $Z(G)^\circ$ on O is trivial. Thus $v(H_\xi)_x = v(H_\xi|_{X_0})_x$ for $x \in X_0, \xi \in \mathfrak{z}(\mathfrak{g})$. In the LHS (resp., RHS) of the previous equality v denotes the Hamiltonian vector field on $O \times X_0$ (resp. $X_0 \cong X_0 \times \{eH^\circ\}$). Thus the functions $H_\xi|_{X_0}, \xi \in \mathfrak{z}(\mathfrak{g})$, satisfy condition (H1). \square

There is the natural embedding $\mathfrak{z}(\mathfrak{g}) = \mathfrak{g}^G \hookrightarrow \mathfrak{g} // G$.

Corollary 4.1.6. *Let G, X be as above. Then*

- (1) $\text{im } \psi_{G,X} \subset \mathfrak{z}(\mathfrak{g})$ and $\psi_{G,X} = \psi_{Z(G)^\circ, X}$.
- (2) $\overline{\text{def}}_G(X) = \overline{\text{def}}_{Z(G)^\circ}(X), \underline{\text{def}}_G(X) = \underline{\text{def}}_{Z(G)^\circ}(X)$.

Proof. The first assertion follows directly from the definitions. To prove the second one we may assume that $X \cong X_0 \times O$ (in the notation of the previous theorem). In this case our assertions are obvious. \square

4.2. The affine case. We preserve the notation of the previous subsection and suppose that X is affine. Recall that G is a connected group such that $G \cong Z(G)^\circ \times (G, G)$.

We have a Hamiltonian open embedding $(X_0 \times O)/\Gamma \hookrightarrow X$ with the image X^0 . Note that X_0 is a closed subvariety in X . In particular, X_0 is affine. Denote by \overline{O} the affine Hamiltonian variety $\text{Spec}(\mathbb{K}[O])$. $X_0 \times O$ is embedded into $X_0 \times \overline{O}$ as an open subset with the complement of codimension not less than 2. Thus we have the action $\Gamma : X_0 \times \overline{O}$ by Hamiltonian automorphisms and the Hamiltonian morphism $\iota : (X \times \overline{O})/\Gamma \rightarrow X$ extending the embedding $(X_0 \times O)/\Gamma \hookrightarrow X$.

Theorem 4.2.1. *The morphism ι defined above is an isomorphism.*

In the proof of the theorem we use the following lemma proved, for example, in [Kr], Section 3.4.

Lemma 4.2.2. *Let X, Y be irreducible affine varieties and $\varphi : X \rightarrow Y$ a birational morphism such that $\text{im } \varphi$ contains an open subset $Y^0 \subset Y$ with $\text{codim}_Y(Y \setminus Y^0) > 1$. Suppose Y is normal. Then φ is an isomorphism.*

of Theorem 4.2.1. It is enough to prove that $\text{codim}_X X \setminus X^0 \geq 2$ (Lemma 4.2.2). In the proof we may replace G with (G, G) and assume that G is semisimple. The proof is in three steps.

Step 1. Let us prove that $\mu_{G,X}|_{\overline{Gx}}$ is a finite morphism for $x \in X^0$. This would imply, in particular, that the closed G -orbit in \overline{Gx} is a point for $x \in X^0$ (and hence, in virtue of the Luna slice theorem, for any $x \in X$).

Put $A = \mathbb{K}[\overline{Gx}]$, $B = \mathbb{K}[\overline{G\eta}]$. The dominant morphism $\mu_{G,X} : \overline{Gx} \rightarrow \overline{G\eta}$ induces the monomorphism $B \hookrightarrow A$. The corresponding extension of the fraction fields $\text{Quot}(B) \subset \text{Quot}(A)$ is finite, its degree is equal to $\#(G_\eta/G_x)$. Denote by $\overline{A}, \overline{B}$ the integral closures of A and B in $\text{Quot}(A)$. Since $\overline{A}, \overline{B}$ are integrally closed in $\text{Quot}(A)$, one gets $\text{Quot}(\overline{A}) = \text{Quot}(\overline{B}) = \text{Quot}(A)$. Moreover, the algebra extensions $A \subset \overline{A}, B \subset \overline{B}$ are finite and $\overline{A}, \overline{B}$ are stable under the action of G on $\text{Quot}(A)$. Put $Z_1 = \text{Spec}(\overline{A})$, $Z_2 = \text{Spec}(\overline{B})$. These are normal G -varieties. We obtain the following commutative diagram

$$\begin{array}{ccc} Z_1 & \xrightarrow{\varphi_1} & Z_2 \\ \downarrow \psi_1 & & \downarrow \psi_2 \\ \overline{Gx} & \xrightarrow{\varphi_2} & \overline{G\eta} \end{array}$$

Here the morphisms ψ_1, ψ_2 are finite and the morphisms ψ_1, φ_1 are birational. Note that both Z_1 and Z_2 contain an open orbit isomorphic to Gx . Being birational and G -equivariant, the morphism φ_1 induces an isomorphism of these orbits. Further, note that the G -variety $\overline{G\eta}$ contains only finitely many orbits and the dimensions of all these orbits are even. Taking into account that ψ_2 is finite, we get $\text{codim}_{Z_2}(Z_2 \setminus Gx) \geq 2$. Lemma 4.2.2 implies that φ_1 is an isomorphism. Thus φ_2 is finite.

Step 2. Here we prove that $\mu_{G,X}^{-1}(0) = X^G$. Clearly, $\mu_{G,X}(X^G) = 0$. Let $x \in \mu_{G,X}^{-1}(0)$. At first, we show that $\mathfrak{g}_x^\perp \subset \mathfrak{g}$ consists of nilpotent elements. Denote by X_1 the stratum of X (see Proposition 2.4.1) containing x . By Example 3.2.4, X_1 is a Hamiltonian subvariety. By the definition of X_1 , $x \in \overline{X_1}^{\max}$. Proposition 3.1.4 implies

$$(4.2) \quad d_x \mu_{G,X_1}(T_x^P X_1) = \mathfrak{g}_x^\perp.$$

Let \mathcal{N} denote the cone in \mathfrak{g} consisting of all nilpotent elements. Note that $\text{im } \mu_{G,X_1} \subset \mathcal{N}$. Thus $\text{im } d_x \mu_{G,X_1}$ coincides with the image of the morphism of the tangent cones $T_x X_1 \rightarrow \mathcal{N}$ induced by μ_{G,X_1} . So $\text{im } d_x \mu_{G,X_1} \subset \mathcal{N}$. Using (4.2), we see that $\mathfrak{g}_x^\perp \subset \mathcal{N}$. By Theorem 1 from [B], Chapter 7, §10, \mathfrak{g}_x is a parabolic subalgebra of \mathfrak{g} . Since G/G_x is quasiaffine, we get $\mathfrak{g}_x = \mathfrak{g}$.

Step 3. Let us complete the proof. Denote by Z an irreducible component of $X \setminus X^0$. This is a G -stable subvariety in X . Denote by η_1 an element from a unique open G -orbit in $\mu_{G,X}(Z)$. The intersection $Z \cap X^G$ is non-empty because $Z \subset X$ is closed and any closed G -orbit consists of one point (step 1). Thus $0 \in \mu_{G,X}(Z)$. Since the dimension of any fiber of a morphism is not less than the dimension of a general one,

$$(4.3) \quad \dim Z \cap \mu_{G,X}^{-1}(0) \geq \dim Z \cap \mu_{G,X}^{-1}(\eta_1) = \dim Z - \dim G\eta_1.$$

On the other hand,

$$(4.4) \quad \dim Z \cap \mu_{G,X}^{-1}(0) \leq \dim \mu_{G,X}^{-1}(0) = \dim X^G,$$

thanks to step 2. By step 1,

$$(4.5) \quad \dim X^G = \dim X//G \leq \dim X - m_G(X) = \dim X - \dim G\eta.$$

It follows from (4.3), (4.4), (4.5) that $\dim Z - \dim G\eta_1 \leq \dim X - \dim G\eta$. Since the dimension of any adjoint orbit is even, $\dim G\eta \geq \dim G\eta_1 + 2$. Therefore $\dim Z \leq \dim X - 2$. \square

Corollary 4.2.3. *Let G be an arbitrary connected reductive group and X an irreducible affine CN Hamiltonian variety. Then:*

- (1) *Any closed orbit for the action $(G, G) : X$ is a point.*
- (2) *Any irreducible component of a fiber of $\pi_{(G, G), X}$ contains a dense (G, G) -orbit.*
- (3) *The restriction of $\pi_{(G, G), X} : X \rightarrow X//G$ to $\mu_{(G, G), X}^{-1}(0)$ is a finite bijective morphism. In particular, if X is normal, then this is an isomorphism.*
- (4) *$\underline{\text{def}}_G(X) = \underline{\text{def}}_{Z(G)^\circ}(X//G)$, $\overline{\text{def}}_G(X) = \overline{\text{def}}_{Z(G)^\circ}(X//G)$. Here $X//G$ is equipped with the structure of a Hamiltonian $Z(G)^\circ$ -variety according to Example 3.2.10.*

Proof. Let us check that $(G, G)x$ is closed iff $x \in X^{(G, G)}$ iff $\mu_{(G, G), X}(x) = 0$. Indeed, to prove this we may replace X with its normalization. Then we are done by the previous theorem.

Now to prove the third assertion it is enough to show that the morphism $\pi_{(G, G), X}|_{X^{(G, G)}}$ is finite. Let \tilde{X} be the normalization of X . Then the natural morphisms $\tilde{X}//G \rightarrow X//G$, $\tilde{X}^{(G, G)} \rightarrow X^{(G, G)}$ are finite and dominant. To prove assertions 2-4 we may assume that X is normal. Also we may assume that G is connected and $G \cong Z(G)^\circ \times (G, G)$. Now our assertions are direct consequences of Theorem 4.2.1 (in the notation of this theorem the Hamiltonian $Z(G)^\circ$ -variety $X//G$ is isomorphic to X_0/Γ). \square

Now we consider the case of a smooth variety X .

Proposition 4.2.4. *Let $G = Z(G)^\circ \times (G, G)$, X be smooth and affine and the action $G : X$ locally effective. In the preceding notation, $(G, G) \cong \text{Sp}(2m_1) \times \dots \times \text{Sp}(2m_k)$, \overline{O} is the direct sum of the tautological $\text{Sp}(2m_k)$ -modules, X_0 is smooth and the action $\Gamma : X_0 \times \overline{O}$ is free.*

Proof. It follows from Theorem 4.1.1 that the morphism $\pi_{\Gamma, X_0 \times \overline{O}} : X_0 \times \overline{O} \rightarrow X$ is étale in codimension 1. By Proposition 3.5.1, $\pi_{\Gamma, X_0 \times \overline{O}}$ is étale. Therefore $X_0 \times \overline{O}$ is smooth and the action of Γ is free. It follows from the Luna slice theorem that \overline{O} is a (G, G) -module. Note that \overline{O} is symplectic as a (G, G) -module. It is enough to prove that if G is simple, then $G \cong \text{Sp}(2m)$ and \overline{O} is the tautological $\text{Sp}(2m)$ -module. In [V1] the list of all linear representations of simple groups possessing a dense orbit is given. Only one of these representations is symplectic. \square

4.3. An estimate on the dimension of a fiber of $\psi_{G, X}$.

Proposition 4.3.1. *Let G be a connected reductive group, X an irreducible CN Hamiltonian G -variety. Then the codimension of any fiber of $\psi_{G, X}$ is not less than $\underline{\text{def}}_G(X)$.*

Proof. Using Corollary 4.1.6, we may replace G with $Z(G)^\circ$ and assume that $G = T$ is a torus. Further, we may assume that X is normal.

To prove the proposition in this case we need three lemmas

Lemma 4.3.2. *Let T be a torus, X an irreducible affine T -variety and Z an irreducible component of a fiber of $\pi_{T, X}$. If the action $T : X$ is locally effective, then so is the action $T : Z$.*

of Lemma 4.3.2. Assume the converse: there exists a non-trivial connected subgroup $T_0 \subset T$ acting trivially on Z . Choose a point $z \in Z$ not lying in another component of $\pi_{T,X}^{-1}(y)$, where $y = \pi_{T,X}(z)$. We see that

$$(4.6) \quad \pi_{T_0,X}^{-1}(\pi_{T_0,X}(z)) \subset \pi_{T,X}^{-1}(y).$$

But the action $T_0 : Z$ is trivial. This yields

$$(4.7) \quad \pi_{T_0,X}^{-1}(\pi_{T_0,X}(z)) \cap Z = \{z\}.$$

It follows from (4.6), (4.7) and the choice of z that $\pi_{T_0,X}^{-1}(\pi_{T_0,X}(z)) = \{z\}$. This implies that any fiber of the quotient morphism $\pi_{T_0,X}$ is trivial. In other words, the action $T_0 : X$ is trivial. Contradiction. \square

Lemma 4.3.3. *Let T be a torus and X a smooth irreducible Hamiltonian T -variety such that the action $T : X$ is locally effective. Then for all $\eta \in \text{im } \mu_{T,X}$ any irreducible component of $\mu_{T,X}^{-1}(\eta)$ contains a point x such that $d_x \mu_{T,X} : T_x X \rightarrow \mathfrak{t}$ is surjective.*

of Lemma 4.3.3. Let x be a point lying in a component Y of $\mu_{T,X}^{-1}(\eta)$ and not lying in another component. There exists a T -stable open affine neighborhood of x in X ([Su]). Thus one may assume that X is affine.

By the choice of x , any irreducible component of $\pi_{T,X}^{-1}(\pi_{T,X}(x))$ containing x is contained in Y . It follows from Lemma 4.3.2 that Y contains a point x_1 such that $\dim T x_1 = \dim T$.

Let X' be a stratum (see Proposition 2.4.1) of X containing x_1 . By Example 3.2.4, X' is a T -stable subvariety of X and the action $T : X'$ is Hamiltonian with the moment map $\mu_{T,X'} = \mu_{T,X}|_{X'}$. Proposition 3.1.4 and the equality $\dim T x_1 = \dim T$ imply that $\mu_{T,X'} = \mu_{T,X}|_{X'}$ is a submersion in x_1 . In particular, $\mu_{T,X}$ is a submersion in x_1 . \square

Lemma 4.3.4. *Let T be a torus, X an irreducible affine T -variety, X_1 a T -stable subvariety of X . Then $\dim X_1 - m_T(X_1) \leq \dim X - m_T(X)$.*

of Lemma 4.3.4. We may assume that the action $T : X$ is locally effective. Let T_1 denote the inefficiency kernel for the action $T : X_1$. Then $\dim X_1 - m_T(X_1) = \dim X_1 - \dim T + \dim T_1 = \dim X - \dim T - (\dim X - \dim T_1 - \dim X_1) \leq \dim X - \dim T - (\dim X // T_1 - \dim X_1)$. It remains to notice that $\dim X_1 = \dim X_1 // T_1 \leq \dim X // T_1$. \square

The proof of the proposition is by induction on $\dim X$. The case $\dim X = 0$ is obvious.

Let T_0 denote the inefficiency kernel for the action $T : X$. Using Remark 3.1.2, we may replace T with T/T_0 and assume that the action $T : X$ is locally effective. Let us choose $\alpha \in \mathfrak{t}$ and prove that $\dim \mu_{T,X}^{-1}(\alpha) \leq \dim X - \dim T$. Choosing a point on $\mu_{T,X}^{-1}(\alpha)$ and replacing X with an invariant affine neighborhood of this point, we may assume that X is affine.

Let $X = X_0 \cup X_1 \cup \dots \cup X_k$ be the stratification introduced in Proposition 2.4.1, where X_0 is an open stratum. By the inductive assumption, $\dim \mu_{G,X}^{-1}(\alpha) \cap X_i \leq \dim X_i - m_T(X_i)$ for $i > 0$. It follows from Lemma 4.3.4 that $\dim X_i - m_T(X_i) \leq \dim X - \dim T$. It remains to show that $\dim \mu_{T,X_0}^{-1}(\alpha) \leq \dim X - \dim T$. By Lemma 4.3.3, for any irreducible component Z of $\mu_{T,X_0}^{-1}(\alpha)$ there exists $z \in Z$ such that μ_{T,X_0} is a submersion in z . This implies $\dim Z = \dim X - \dim T$. \square

4.4. On the structure of $C_{G,X}, \tau_{G,X}$. Put $\mathfrak{a}_{G,X} = \overline{\text{im } \psi_{G,X}}$. The space $\mathfrak{z}(\mathfrak{g}) \cong \mathfrak{g}^G$ is naturally embedded into \mathfrak{g}/G . Since X is CN, $\mathfrak{a}_{G,X}$ lies in $\mathfrak{z}(\mathfrak{g})$ and coincides with $\overline{\text{im } \mu_{Z(G)^\circ, X}} = \mathfrak{a}_{Z(G)^\circ, X}$. Denote by $\tau_{G,X}^1$ the embedding $\mathfrak{a}_{G,X} \rightarrow \mathfrak{g}/G$ and by $\tau_{G,X}^2 : C_{G,X} \rightarrow \mathfrak{a}_{G,X}$ a unique morphism such that $\tau_{G,X} = \tau_{G,X}^1 \circ \tau_{G,X}^2$.

Proposition 4.4.1. *Let \mathfrak{t}_0 be the Lie algebra of the inefficiency kernel of the action $Z(G) : X$. Suppose X is equidefectinal. Then $\mathfrak{a}_{G,X}$ is an affine subspace of $\mathfrak{z}(\mathfrak{g})$ of dimension $\text{def}_G(X)$ intersecting \mathfrak{t}_0 in a unique point.*

Proof. The claim on the dimension of $\mathfrak{a}_{G,X}$ is obvious. By Corollary 4.1.6, $\text{def}_G(X) = \text{def}_{Z(G)^\circ}(X)$. Therefore we may replace G with $Z(G)^\circ$ and assume that $G = T$ is a torus. Denote by T_0 the connected subgroup of T corresponding to \mathfrak{t}_0 . Let ξ be a point in $\overline{\text{im } \mu_{T_0, X}}$ and Z be an irreducible component of $\mu_{T_0, X}^{-1}(\xi)$. Since T_0 acts trivially on X , $\mu_{T_0, X}^*(\mathbb{K}[\mathfrak{t}_0])$ lies in the center of $\mathbb{K}(X)$. Thus Z is a component of a Poisson subvariety of X . Proposition 2.1.3 implies that $Z \subset X$ is a Poisson subvariety. Since T is connected, any Poisson subvariety in X is Hamiltonian (see Example 3.2.4). For $\xi \in \overline{\text{im } \mu_{T_0, X}}$ in general position we get $\text{def}_T(Z) = m_T(Z) = \dim T - \dim T_0 = \text{def}_T(X)$. Thus $\text{def}_T(X) = \text{def}_T(Z) + \dim \overline{\text{im } \mu_{T_0, X}} \geq \text{def}_T(X) + \dim \overline{\text{im } \mu_{T_0, X}}$. This implies that $\text{im } \mu_{T_0, X}$ is a point. Note that this point is the (orthogonal) projection of $\text{im } \mu_{T, X}$ to \mathfrak{t}_0 . Hence $\text{im } \mu_{T, X}$ is contained in an affine subspace in \mathfrak{t} of dimension $\text{def}_T(X)$ intersecting \mathfrak{t}_0 in a unique point. Comparing the dimensions, we see that $\overline{\text{im } \mu_{T, X}}$ coincides with this affine space. \square

The following proposition is the main result of this subsection.

Proposition 4.4.2. *Let G be a connected reductive group, X a normal irreducible equidefectinal CN Hamiltonian G -variety. Then $\text{im } \tilde{\psi}_{G,X}$ is an open subset of $C_{G,X}$ and the restriction of $\tau_{G,X}^2 : C_{G,X} \rightarrow \mathfrak{a}_{G,X}$ to $\text{im } \tilde{\psi}_{G,X}$ is étale.*

Proof. Recall that, by definition, $C_{G,X}$ is a normal variety. Since $\overline{\text{def}}_G(X) = \text{def}_G(X)$, the morphism $\tilde{\psi}_{G,X}$ is equidimensional by Proposition 4.3.1. An equidimensional morphism to a normal variety is open (see [Ch]). In particular, $\text{im } \tilde{\psi}_{G,X}$ is an open subset of $C_{G,X}$.

Since G is connected, $\mathbb{K}(X)^G$ is algebraically closed in $\mathbb{K}(X)$. Therefore $\mathbb{K}[C_{G,X}]$ is an integral closure of $\psi_{G,X}^*(\mathbb{K}[\mathfrak{g}]^G)$ in $\mathbb{K}(X)$. It follows from Corollary 4.1.6 that $\psi_{G,X}^*(\mathbb{K}[\mathfrak{g}]^G) = \psi_{Z(G)^\circ, X}^*(\mathbb{K}[\mathfrak{z}(\mathfrak{g})])$. In other words, the varieties $C_{G,X}$ and $C_{Z(G)^\circ, X}$ are naturally isomorphic and $\tilde{\psi}_{G,X} = \tilde{\psi}_{Z(G)^\circ, X}$, $\tau_{G,X} = \tau_{Z(G)^\circ, X}$. By the definitions of $\mathfrak{a}_{G,X}, \tau_{G,X}^2$, we have $\mathfrak{a}_{G,X} = \mathfrak{a}_{Z(G)^\circ, X}$, $\tau_{G,X}^2 = \tau_{Z(G)^\circ, X}^2$. Therefore we may assume that $G = T$ is a torus.

Denote by T_0 the inefficiency kernel for the action $T : X$. Since X is equidefectinal, $\dim \mathfrak{a}_{T,X} = \dim T/T_0$. Thus $\psi_{T,X}^*(\mathbb{K}[\mathfrak{t}]) = \psi_{T/T_0, X}^*(\mathbb{K}[\mathfrak{t}/\mathfrak{t}_0])$, in other words, $C_{T,X}$ and $C_{T/T_0, X}$ are naturally isomorphic. By Proposition 4.4.1, the restriction of the projection $\mathfrak{t} \rightarrow \mathfrak{t}/\mathfrak{t}_0$ to $\mathfrak{a}_{T,X}$ is an isomorphism. Clearly, we have the commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & C_{T,X} & \xrightarrow{\quad} & \mathfrak{a}_{T,X} \\ \downarrow & \nearrow \cong & & & \downarrow \cong \\ C_{T/T_0, X} & \xrightarrow{\quad} & \mathfrak{t}/\mathfrak{t}_0 \cong \mathfrak{a}_{T/T_0, X} & & \end{array}$$

Therefore we may assume that the action $T : X$ is (locally) effective. Let us show now that $\tau_{T,X} : C_{T,X} \rightarrow \mathfrak{t}$ is étale in points of $C_{T,X}^{\text{reg}} \cap \tilde{\psi}_{T,X}(X^{\text{reg}})$. Indeed, for any $y \in C_{T,X}^{\text{reg}}$ and

any $x \in X^{reg} \cap \tilde{\psi}_{T,X}^{-1}(y)$ we have $d_x \mu_{T,X} = d_y \tau_{T,X} \circ d_x \tilde{\psi}_{T,X}$. By Lemma 4.3.3, there exists a point $x \in X^{reg} \cap \tilde{\psi}_{T,X}^{-1}(y)$ such that $d_x \mu_{T,X}$ is a surjection. Thus $\tau_{G,X}$ is étale in y .

Now we check that $C_{T,X}^{reg} \cap \tilde{\psi}_{T,X}(X^{reg})$ is an open subset in $\text{im } \tilde{\psi}_{T,X}$, whose complement is of codimension not less than 2. Indeed, $\text{codim}_{C_{T,X}} C_{T,X}^{sing}, \text{codim}_X X^{sing} \geq 2$ because $X, C_{T,X}$ are normal. Since $\tilde{\psi}_{G,X}$ is equidimensional (Proposition 4.3.1), $\text{codim}_{\text{im } \tilde{\psi}_{T,X}} \text{im } \tilde{\psi}_{T,X} \setminus \tilde{\psi}_{T,X}(X^{reg}) \geq 2$. This shows our claim.

To complete the proof of the proposition it is enough to apply Proposition 3.5.1 to the morphism $\text{im } \tilde{\psi}_{T,X} \rightarrow \mathfrak{t}$. \square

4.5. The proof of Theorem 1.2.9 for CN varieties. In this subsection we suppose that X is normal, affine and strongly equidefectinal. In the proof of theorem 1.2.9 we may assume that $G \cong Z(G)^\circ \times (G, G)$.

Proposition 4.5.1. *Let T be a torus and X be irreducible affine strongly equidefectinal Hamiltonian T -variety. Then the action $T : X$ is stable.*

Proof. We may assume that the action $T : X$ is effective. Indeed, let T_0 be the inefficiency kernel. Since X is equidefectinal, the hamiltonians $H_\xi, \xi \in \mathfrak{t}_0$, are constant. Thus X is strongly equidefectinal also as a Hamiltonian T/T_0 -variety (with the same stratification as for the action $T : X$).

Let $X = \coprod_i X_i$ be a stratification given by Definition 1.2.8. Let us show that there exists a stratum X_i such that $\text{def}_T(X_i) = \dim T$ and there is a closed T -orbit in X_i . Indeed, otherwise all closed T -orbits lie in $\bigcup_{i \in J} X_i$, where $J = \{i \mid \text{def}_T(X_i) < \dim T\}$. In other words, $X//T = \bigcup_{i \in J} \overline{X_i//T}$. Thus there exists $i \in J$ such that $\overline{X_i//T} = \overline{X//T}$. But $\dim \mu_{T,X}(\overline{X_i}) = \text{def}_T(X_i) < \dim \overline{\text{im } \mu_{T,X}}$. Therefore $\mu_{T,X}//T(\overline{X_i//T}) \neq \mu_{T,X}//T(X//T)$. Contradiction. So there is a point $x \in X_i$, where $\text{def}_T(X_i) = \dim T$, such that the orbit Tx is closed.

The action $T : X$ is stable iff there is a closed orbit of dimension $\dim T$, see [Pop1]. It is enough to show that one can find such an orbit even in $\overline{X_i}$. So we may assume that $x \in X^{max}$.

Let us prove that the action of $T_0 := (T_x)^\circ$ on $T_x X / T_x^P X$ is trivial. Assume the converse. Let us choose a T_0 -stable complement V to $T_x^P X$ in $T_x X$. It follows from the Luna slice theorem that there is a T_0 -stable smooth locally-closed subvariety $Y \subset X$ such that $x \in Y, T_x Y = V$. Replacing Y with some open subset we may assume that $T_y^P X \oplus T_y Y = T_y X$ for any $y \in Y$. By the choice of Y , $\xi_* y \in T_y Y$ for any $\xi \in \mathfrak{t}_0$. But since the action $T_x : X$ is Hamiltonian, $\xi_* y \in T_y^P X$ for any $\xi \in \mathfrak{t}_0$. Thus the action $T_0 : Y$ is trivial.

Applying the slice theorem again, we see that it is enough to show that the action $T_0 : T_x X / \mathfrak{t}_* x$ is stable. Since T_0 acts trivially on $T_x X / T_x^P X, \mathfrak{t}_* x$ we reduce to the proof of the stability of the action $T_0 : T_x^P X$.

Let us prove that an action of a torus T on a symplectic T -module U is stable. Indeed, we may assume that the action is effective. Choose linearly independent weights $\lambda_1, \dots, \lambda_k, k = \dim T$, of the T -module U . Since $U \cong U^*$, we see that $-\lambda_1, \dots, -\lambda_k$ are also weights of U . Choose nonzero weight vectors $v_{\lambda_1}, \dots, v_{\lambda_k}, v_{-\lambda_1}, \dots, v_{-\lambda_k}$ in the corresponding weight subspaces. The orbit of $\sum_i (v_{\lambda_i} + v_{-\lambda_i})$ is closed and its dimension is equal to $\dim T$.

Since $T_x^P X$ is a symplectic T_0 -module, we are done. \square

Let X_0, η, Γ be such as in Subsection 4.1, \overline{O} such as in Subsection 4.2.

Lemma 4.5.2. *The Hamiltonian action $Z(G)^\circ : X_0$ is strongly equidefectinal.*

Proof. The Hamiltonian action $G : X_0 \times O$ is strongly equidefectinal because there is an étale Hamiltonian morphism $X_0 \times O \rightarrow X$ (see Lemma 3.4.5, Theorem 4.1.1). Note that any G -stable locally closed subvariety $Y \subset X_0 \times O$ has the form $Y_0 \times O$ for some locally closed $Z(G)^\circ$ -stable subvariety $Y_0 \subset X_0$. Clearly, Y is a Hamiltonian subvariety of X iff Y_0 is a Hamiltonian subvariety of X_0 . If Y is irreducible, then $\underline{\text{def}}_G(Y) = \underline{\text{def}}_{Z(G)^\circ}(Y_0)$, $\overline{\text{def}}_G(Y) = \overline{\text{def}}_{Z(G)^\circ}(Y_0)$. The intersections of the strata of X with X_0 form a stratification of X_0 satisfying the assumptions of Definition 1.2.8. \square

of Theorem 1.2.9 in the CN case. By Theorem 4.2.1, $X \cong (X_0 \times \overline{O})/\Gamma$. We easily reduce to the case $X = X_0 \times \overline{O}$. Clearly, X satisfies condition (b) or (c) of the second assertion iff $\eta = 0$ iff \overline{O} is a point. By Lemma 4.5.2, a Hamiltonian $Z(G)^\circ$ -variety X_0 is strongly equidefectinal. Proposition 4.5.1 implies that the action $Z(G)^\circ : X_0$ is stable. This completes the proof because \overline{O} contains the dense (G, G) -orbit O . \square

5. REDUCTION TO THE CENTRAL-NILPOTENT CASE

This is the most important section of the paper. Here we show how to reduce the proofs of the theorems stated in the Introduction to the case of CN varieties. Throughout the section G denotes a reductive group and X a quasiprojective Hamiltonian G -variety, if otherwise is not stated.

The first subsection is devoted to an algebraic version of the local-cross section theorem of Guillemin and Sternberg, see [GS]. The original theorem deals with Hamiltonian actions of compact groups on real manifolds. Knop in [Kn3] proved an analog of this theorem for Hamiltonian actions of reductive algebraic groups on symplectic varieties. Our approach is based on his.

Let us explain what we mean by a cross-section. Suppose that X is quasi-projective and normal. Fix a Levi subgroup $L \subset G$. In general, the subvariety $\mu_{G,X}^{-1}(\mathfrak{l}) \subset X$ is $N_G(L)$ -stable but the action of $N_G(L)$ is not Hamiltonian. However, there is an $N_G(L)$ -stable open subset $\mathfrak{l}^{pr} \subset \mathfrak{l}$ such that the $N_G(L)$ -subscheme $\tilde{Y} = \mu_{G,X}^{-1}(\mathfrak{l}^{pr})$ is normal (Corollary 5.1.3), has a natural Hamiltonian structure with the moment map $\mu_{G,X}|_{\tilde{Y}}$ (Proposition 5.1.4) and the natural morphism $G *_{{N_G(L)}} \tilde{Y} \rightarrow X$ is étale (Corollary 5.1.3).

In Subsection 5.2 we use the construction of the previous subsection for some special choice of L . Namely, take for L the centralizer of $\mu_{G,X}(x)_s$ for $x \in X$ in general position. The variety $\tilde{Y} = \mu_{G,X}^{-1}(\mathfrak{l}^{pr})$ is $N_G(L)$ -irreducible and the natural morphism $G *_{{N_G(L)}} \tilde{Y} \rightarrow X$ is an open embedding (Proposition 5.2.2). The $N_G(L)$ -Hamiltonian variety \tilde{Y} is CN. Moreover, \tilde{Y} is affine provided so is X . Choose an irreducible (=connected) component Y of \tilde{Y} . Put $\mathfrak{a}_{G,X}^{(Y)} = \mathfrak{a}_{N_G(L,Y),Y}$ (see Subsection 4.4) and $W_{G,X}^{(Y)} = N_G(L,Y)/L$. In the case when X is affine and equidefectinal these definitions coincide with those given in the Introduction. At the end of Subsection 5.2 we introduce the factorization of the morphism $\tau_{G,X} = \tau_{G,X}^1 \circ \tau_{G,X}^2$, see the Introduction.

In Subsection 5.3. we prove Theorem 1.2.9. The key step is to reduce the proof for the pair (G, X) to the proof for the pair (L, Y) , where L, Y are such as in the previous paragraph.

Unfortunately, this trick does not work for Theorems 1.2.3, 1.2.5. For example, Y does not intersect some fibers of $\psi_{G,X}$. In Subsections 5.4-5.6 we construct another CN Hamiltonian L -variety R from Y .

In Subsection 5.4 we recall some (mostly standard) properties of H -invariants of an affine G -varieties, where H is the unipotent radical or the derived subgroup of a parabolic subgroup of G .

To construct R we need some parabolic subgroup $P \subset G$ with Levi subgroup L . If we want R to have good properties, we should make some special choice of P . In Subsection 5.5 we establish the notion of a parabolic subgroup P compatible with Y and study the key property of such a subgroup (Proposition 5.5.1).

Subsection 5.6 is devoted to the construction of R . A sketch of the construction was given in Subsection 1.3.

In Subsection 5.7 the basic properties of R are studied. The next subsection is devoted to the proofs of Theorems 1.2.3, 1.2.5. Here we introduce some good action of $W_{G,X}^{(Y)}$ on the quotient $R//L$ (Lemma 5.8.1, Proposition 5.8.2). Notice that there is no natural action of $N_G(L, Y)$ on R .

In Subsection 5.9 we establish a relation between $\mathbb{K}[C_{G,X}]$ and $\mathbb{K}[X]^G \cap \mathfrak{z}(\mathbb{K}(X)^G)$ (Proposition 5.9.1) and prove Theorem 1.2.7. In Subsection 5.10 we give an example of a connected group G and a conical symplectic affine variety X such that the group $W_{G,X}^{(Y)}$ is not generated by reflections.

5.1. Local cross-sections. Let \mathfrak{l} be a Levi subalgebra in \mathfrak{g} . Put $N = N_G(\mathfrak{l})$.

Definition 5.1.1. An element $\xi \in \mathfrak{l}$ is called *principal* if $\mathfrak{z}_{\mathfrak{g}}(\xi_s) \subset \mathfrak{l}$. The subset of \mathfrak{l} consisting of all principal elements is denoted by \mathfrak{l}^{pr} .

Clearly, \mathfrak{l}^{pr} is a nonempty open subset of \mathfrak{l} . There is another equivalent definition of $\mathfrak{l}^{pr} \subset \mathfrak{l}$. Namely, consider a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{l}$ and the root systems $\Delta(\mathfrak{g}), \Delta(\mathfrak{l})$ of $\mathfrak{g}, \mathfrak{l}$, respectively, associated with \mathfrak{t} . The subset $\mathfrak{l}^{pr} \subset \mathfrak{l}$ consists precisely of elements $\xi \in \mathfrak{l}$ such that the semisimple part ξ_s of ξ is conjugate under the action of N to some element of \mathfrak{t}^0 , where

$$(5.1) \quad \mathfrak{t}^0 := \mathfrak{t} \setminus \bigcup_{\alpha \in \Delta(\mathfrak{g}) \setminus \Delta(\mathfrak{l})} \ker \alpha.$$

It follows from this definition and the Chevalley restriction theorem that there is $f \in \mathbb{K}[\mathfrak{l}]^N$ such that $\mathfrak{l}^{pr} = \{\xi \in \mathfrak{l} \mid f(\xi) \neq 0\}$.

Consider the subscheme $\tilde{Y} := \mu_{G,X}^{-1}(\mathfrak{l}^{pr}) \subset X$. This is a principal N -stable open subscheme in $\mu_{G,X}^{-1}(\mathfrak{l})$. In particular, if X is affine, then so is \tilde{Y} . Let us describe some properties of the natural morphism $\varphi : G *_N \tilde{Y} \rightarrow X$.

Proposition 5.1.2. *The morphism $\varphi : G *_N \tilde{Y} \rightarrow X, [g, y] \mapsto gy$, is non-ramified. Its restriction to any irreducible component of $G *_N \tilde{Y}$ is dominant. $\text{im } \varphi = \psi_{G,X}^{-1}(Z)$ for some open subset $Z \subset \mathfrak{g}//G$.*

Proof. The dimension of any irreducible component of $\mu_{G,X}^{-1}(\mathfrak{l})$ is not less than $\dim X - \dim \mathfrak{g} + \dim \mathfrak{l}$. Thus the dimension of any component of $G *_N \tilde{Y}$ is not less than $\dim X$.

Now we shall check that φ is non-ramified, i.e. that for any $y \in \tilde{Y}$ the linear map $d_y \varphi : T_y(G *_N \tilde{Y}) \rightarrow T_{\varphi(y)} X$ is injective.

We have the natural identification $T_y(G *_N \tilde{Y}) \cong T_y \tilde{Y} \oplus \mathfrak{l}_*^\perp y$. The restriction of φ to \tilde{Y} is the embedding $\tilde{Y} \hookrightarrow X$. Thus $d_y \varphi|_{T_y \tilde{Y}}$ is an embedding. It remains to show that $\xi \in \mathfrak{l}$

provided $\xi_*y \in T_y\tilde{Y}$, $\xi \in \mathfrak{g}$. Let $T_y^P X$, ω_y be such as in Subsection 2.2. For any $\eta \in \mathfrak{l}^\perp$ we have

$$0 = \partial_{\xi_*y} H_\eta(y) = \omega_y(\xi_*y, \eta_*y) = \{H_\xi, H_\eta\}(y) = H_{[\xi, \eta]}(y) = ([\mu_{G,X}(y), \xi], \eta).$$

Therefore $[\xi, \mu_{G,X}(y)] \in \mathfrak{l}$. Let ξ' denote the projection of ξ to \mathfrak{l}^\perp . Recall that $\mu_{G,X}(y) \in \mathfrak{p}^r$. Thus $\xi' \in \mathfrak{z}_{\mathfrak{g}}(\mu_{G,X}(y)) \subset \mathfrak{z}_{\mathfrak{g}}(\mu_{G,X}(y)_s) \subset \mathfrak{l}$ whence $\xi \in \mathfrak{l}$.

So φ is non-ramified. Comparing the dimensions, we see that the restriction of φ to any component of $G *_N \tilde{Y}$ is dominant.

Now we prove the claim on the image of φ . By the alternative description of \mathfrak{p}^r in the beginning of this subsection,

$$(5.2) \quad x \in \text{im } \varphi \Leftrightarrow G\mu_{G,X}(x)_s \cap \mathfrak{t}^0 \neq \emptyset.$$

Put $Z = \mathfrak{t}/W \setminus \pi_{W,\mathfrak{t}}(\mathfrak{t} \setminus \mathfrak{t}^0)$, where $W = N_G(\mathfrak{t})/Z_G(\mathfrak{t})$. Obviously, Z is an open subvariety in $\mathfrak{t}/W \cong \mathfrak{g}/G$. It follows from (5.2) that $\text{im } \varphi = \psi_{G,X}^{-1}(Z)$. \square

Corollary 5.1.3. *If X is normal, then $\varphi : G *_N \tilde{Y} \rightarrow X$ is étale and the scheme \tilde{Y} is normal.*

Proof. φ is étale because this is a non-ramified dominant morphism to a normal variety (see [Mi], Ch.1, Theorem 3.20). Thus the scheme $G *_N \tilde{Y}$ is a normal variety. By Lemma 4.1.3, \tilde{Y} is a normal scheme. \square

We want to equip the N -variety \tilde{Y} with a natural Hamiltonian structure.

Let $y \in \tilde{Y}$. We have the natural isomorphism $T_y X \cong T_y \tilde{Y} \oplus \mathfrak{l}_*^\perp y$. The differential $d_y \mu_{G,X}$ induces the isomorphism $\mathfrak{l}_*^\perp y \cong \mathfrak{l}_*^\perp \mu_{G,X}(y)$. The restriction of the Kostant-Kirillov form ω on the orbit $G\mu_{G,X}(y)$ (see Example 2.3.1) to the subspace $V = \mathfrak{l}_*^\perp \mu_{G,X}(y) \subset \mathfrak{g}_* \mu_{G,X}(y)$ is nondegenerate. Indeed, the subspaces $V, \mathfrak{l}_* \mu_{G,X}(y) \subset \mathfrak{g}_* \mu_{G,X}(y)$ are orthogonal with respect to this form. The form $\omega|_V$ induces the bivector $P_y^{\mu_{G,X}} \in \bigwedge^2 V$ (see Subsection 2.2).

Using the identification $\mathfrak{l}_*^\perp y \cong V$, we obtain the bivector $\tilde{P}_y^{\mu_{G,X}} \in \bigwedge^2 \mathfrak{l}_*^\perp y$. For $n \in N$ there is the natural isomorphism $n_* : \mathfrak{l}_*^\perp y \rightarrow \mathfrak{l}_*^\perp (ny)$. Since $\mu|_{\tilde{Y}}$ is N -equivariant,

$$(5.3) \quad n_* \tilde{P}_y^{\mu_{G,X}} = \tilde{P}_{ny}^{\mu_{G,X}}.$$

For $f, g \in O_{X,y}$ define an element $\{f, g\}^Y \in O_{\tilde{Y},y}$ by

$$(5.4) \quad \{f, g\}^Y(y_1) = \{f, g\}(y_1) - \tilde{P}_{y_1}^{\mu_{G,X}}(df \wedge dg), y_1 \in \tilde{Y}.$$

Proposition 5.1.4. *Let $f, g \in O_{X,y}$. The element $\{f, g\}^Y \in O_{\tilde{Y},y}$ depends only on the restrictions of f, g to \tilde{Y} . Furthermore, $\{\cdot, \cdot\}^Y : \mathbb{K}(\tilde{Y}) \otimes \mathbb{K}(\tilde{Y}) \rightarrow \mathbb{K}(\tilde{Y})$ is an N -invariant Poisson bracket. The action $N : \tilde{Y}$ is Hamiltonian with the moment map $\mu_{N,\tilde{Y}} = \mu_{G,X}|_{\tilde{Y}}$.*

Proof. Let Y_0 be an irreducible component of \tilde{Y} . By Proposition 5.1.2, $\varphi(G *_N (NY_0)) = GY_0$ is dense in X . Hence $Y_0 \cap X^{max} \neq \emptyset$, so we may replace X with X^{max} . Corollary 5.1.3 implies that $G *_N \tilde{Y}$ is smooth. Thus \tilde{Y} is smooth. Consider the distribution $T^P X$ and the "2-form" $\omega \in H^0(X, \bigwedge^2(T^P X)^*)$. By Proposition 2.2.2, $T^P X$ is involutory and ω satisfies (2.8). Let $z \in \tilde{Y}$. Choose $\xi \in \mathfrak{l}^\perp, \eta \in T_z \tilde{Y} \cap T_z^P X$. Since $H_\xi|_{\tilde{Y}} = 0$, we have $\omega_z(\xi_*z, \eta) = \partial_\eta H_\xi(z) = 0$. So $T_z^P X \cap T_z \tilde{Y}$ and $\mathfrak{l}_*^\perp z$ are orthogonal with respect to ω_z . It follows that the bivector $P_z^Y := P_z - \tilde{P}_z^{\mu_{G,X}}$ lies in $\bigwedge^2 T_z \tilde{Y}$, equivalently, the bracket on \tilde{Y} is well-defined. Note that

$$(5.5) \quad T_z^{PY} \tilde{Y} = T_z^P X \cap T_z \tilde{Y}, \mathfrak{g}_* z \subset T_z^P X.$$

Lemma 5.1.5. *Let $H \subset G$ be algebraic groups, Y a quasi-projective H -variety and \tilde{V} a G -stable distribution on $G *_H Y$ such that $\mathfrak{g}_*[e, y] \in \tilde{V}_{[e, y]}$ for all $y \in Y$. Put $V_y = \tilde{V}_{[e, y]} \cap T_y Y$. If \tilde{V} is involutory, then so is V .*

of Lemma 5.1.5. There is a quasi-section (see [PV2], §2) Z for the action $H : G$ by the right translations such that the natural morphism $Z \rightarrow G/H$ is étale. So $\varphi : Y \times Z \rightarrow G *_H Y$, $(y, z) \mapsto [z, y]$, is an étale morphism. The distribution $\varphi^* \tilde{V}$ coincides with $V \otimes TZ \subset T(Y \times Z)$. \square

According to Lemma 5.1.5, $T_z^{P^Y} \tilde{Y}$ is an involutory distribution. Let $\omega^Y \in H^0(\tilde{Y}, \bigwedge^2(T^{P^Y} \tilde{Y})^*)$ be the "2-form" corresponding to P^Y . The form ω^Y is the restriction of ω to $T^{P^Y} \tilde{Y}$. Therefore ω^Y satisfies (2.8). Applying Proposition 2.2.2, we see that P^Y is a Poisson bivector. Note that for $f \in O_{X, y}$ the vector $v_y(f|_{\tilde{Y}})$ coincides with the projection of $v_y(f)$ to $T_y \tilde{Y}$ with respect to the decomposition $T_y X = T_y \tilde{Y} \oplus \mathfrak{l}_*^\perp y$. In particular, for $\xi \in \mathfrak{l}$ we have $v(H_\xi|_{\tilde{Y}}) = \xi_*$ because \tilde{Y} is L -stable. Thus $\mu_{G, X}|_{\tilde{Y}} : \tilde{Y} \rightarrow \mathfrak{l}$ satisfies the axioms of a moment map. \square

The variety \tilde{Y} equipped with this Hamiltonian action of L is denoted by $\text{Red}_G^N(X)$. Let us investigate functorial properties of $\text{Red}_G^N(\cdot)$.

Proposition 5.1.6. *Let X_1, X_2 be normal quasiprojective Hamiltonian G -varieties and $\varphi : X_1 \rightarrow X_2$ a Hamiltonian morphism. Then $\varphi(\text{Red}_G^N(X_1)) \subset \text{Red}_G^N(X_2)$. Let $\text{Red}_G^N(\varphi) : \text{Red}_G^N(X_1) \rightarrow \text{Red}_G^N(X_2)$ be the corresponding morphism. This is a Hamiltonian N -morphism.*

Proof. The only not obvious thing here is that $\text{Red}_G^N(\varphi)$ is a Poisson morphism. Let $\{\cdot, \cdot\}^{Y_i} : \mathbb{K}(X_i) \otimes \mathbb{K}(X_i) \rightarrow \mathbb{K}(\text{Red}_G^N(X_i))$, $i = 1, 2$, be the brackets defined by (5.4). We have to prove that for $y \in \text{Red}_G^N(X_1)$ and $f, g \in O_{X_2, \varphi(y)}$ there is the equality

$$(5.6) \quad \{\varphi^* f, \varphi^* g\}^{Y_1}(y) = \{f, g\}^{Y_2}(\varphi(y)).$$

Consider the bivectors $\tilde{P}_y^{\mu_{G, X_1}}$ and $\tilde{P}_{\varphi(y)}^{\mu_{G, X_2}}$. Since φ is a Hamiltonian morphism, we see that $d_y \Phi \tilde{P}_y^{\mu_{G, X_1}} = P_{\varphi(y)}^{\mu_{G, X_2}}$. The equality (5.6) follows from the definitions of $\{\cdot, \cdot\}^{Y_i}$, because $\varphi : X_1 \rightarrow X_2$ is a Poisson morphism. \square

Any irreducible component of $\text{Red}_G^N(X)$ is a Hamiltonian L -variety.

Proposition 5.1.7. $m_L(Y) = m_G(X) - \dim G + \dim L$, $\overline{\text{def}}_G(X) = \overline{\text{def}}_L(Y)$, $\underline{\text{def}}_G(X) = \underline{\text{def}}_L(Y)$ for any irreducible component $Y \subset \tilde{Y}$.

Proof. Since $\varphi : G *_L Y \rightarrow X$ is dominant and non-ramified, $m_G(X) = \dim G/L + m_L(Y)$ and GY is dense in X . The latter implies the equality of the upper defects. Taking into account that $\text{im } \mu_{L, Y} \subset \mathfrak{l}^{pr}$, we get $\text{rk}_L(Y) = \text{rk}_G(X) - \dim G/N$. This implies $\underline{\text{def}}_G(X) = \underline{\text{def}}_L(Y)$. \square

5.2. $\mathfrak{a}_{G, X}^{(Y)}$ and $W_{G, X}^{(Y)}$. We preserve the notation of the previous subsection. Suppose that X is irreducible and normal. By Corollary 5.1.3, if $\text{Red}_G^N(X)$ is non-empty, then φ is a dominant étale Hamiltonian morphism and $\text{Red}_G^N(X)$ is a normal (possibly non-connected) Hamiltonian N -variety. We want to show that for a special choice of L the morphism φ is an open embedding.

Namely, let $L \subset G$ be the centralizer in G° of $\mu_{G, X}(x)_s$ for $x \in X$ in general position. In other words, L is the stabilizer of the closed orbit in general position for the action

$G^\circ : \overline{\text{im } \mu_{G,X}}$ (so-called, principal isotropy group, see [PV2], Theorem 7.12). Notice that L is defined uniquely up to G° -conjugacy.

Definition 5.2.1. Such a subgroup L is called the *principal centralizer* of X .

Proposition 5.2.2. (1) *The morphism $\varphi : G *_N \text{Red}_G^N(X) \rightarrow X$ is an open embedding. In particular, $\text{Red}_G^N(X)$ is N -irreducible.*
 (2) *If X is, in addition, affine, then the natural morphism $\text{Red}_G^N(X)//N \rightarrow X//G$ is an open embedding.*

Proof. Let us check assertion 1. Since φ is étale, it remains to prove that φ is injective. Let $y_1, y_2 \in \text{Red}_G^N(X)$, $g_1, g_2 \in G$ be such that $g_1 y_1 = g_2 y_2$. We may assume $g_1 = 1$. Note that $\mu_{G,X}(y_1)_s = \text{Ad}(g_2) \mu_{G,X}(y_2)_s$. The centralizer of $\mu_{G,X}(y_i)_s$ in \mathfrak{g} coincides with \mathfrak{l} . Thus $g_2 \in N$. Since φ is an embedding, we see that $G *_N \text{Red}_G^N(X)$ is irreducible. By Lemma 4.1.3, $\text{Red}_G^N(X)$ is N -irreducible.

Recall that $\text{im } \varphi = \psi_{G,X}^{-1}(Z)$, where Z is an open subset of $\mathfrak{g}//G$. Thus $G *_N \text{Red}_G^N(X) \cong \text{im } \varphi = \pi_{G,X}^{-1}(\text{im } \varphi//G)$. This implies assertion 3. \square

By the choice of L , for any $y \in \text{Red}_G^N(X)$ the centralizer of $\mu_{L, \text{Red}_G^N(X)}(y)_s$ in \mathfrak{g} coincides with \mathfrak{l} . Thus $\text{Red}_G^N(X)$ is a CN Hamiltonian N -variety.

Choose a component $Y \subset \text{Red}_G^N(X)$ and put $N_0 = N_G(L, Y)$. As we have seen in the previous subsection, Y is a normal CN Hamiltonian N_0 -variety with the moment map $\mu_{N_0,Y} = \mu_{G,X}|_Y$. We say that Y is an L -cross-section of X . If X is affine, then so is Y . It follows from Proposition 5.2.2 that the triple (L, Y, N_0) is determined uniquely up to G -conjugacy and the natural map $G *_N Y \rightarrow X$ is an open embedding, whose image equals to $\psi_{G,X}^{-1}(Z)$ for some open subset $Z \subset \mathfrak{g}//G$.

Put $\mathfrak{a}_{G,X}^{(Y)} = \mathfrak{a}_{L,Y}$ (see Subsection 4.4).

Remark 5.2.3. Suppose X is equidefectinal. Then so is Y (Proposition 5.1.7). By Proposition 4.4.1, $\mathfrak{a}_{G,X}^{(Y)}$ is an affine subspace in $\mathfrak{z}(\mathfrak{l})$. It follows directly from the definition of $\mathfrak{a}_{G,X}^{(Y)}$ that $\mathfrak{a}_{G,X}^{(Y)} = \mathfrak{a}_{L,Y} = \overline{\pi(\mu_{G,X}(Y))}$, where π is the projection $\mathfrak{l} \rightarrow \mathfrak{l}/[\mathfrak{l}, \mathfrak{l}] \cong \mathfrak{z}(\mathfrak{l})$. So the definition of $\mathfrak{a}_{G,X}^{(Y)}$ given here coincides with that from the Introduction.

Put $W_{G,X}^{(Y)} = N_0/L$. Since Y is a Hamiltonian N_0 -variety, $\mathfrak{a}_{G,X}^{(Y)} \subset \mathfrak{z}(\mathfrak{l})$ is stable under the natural action of $W_{G,X}^{(Y)}$ on $\mathfrak{z}(\mathfrak{l})$.

Clearly, the restriction of $f \in \mathbb{K}[\mathfrak{g}]^G$ to $\mathfrak{a}_{G,X}^{(Y)} \subset \mathfrak{g}$ is $W_{G,X}^{(Y)}$ -invariant. So we have the natural morphism $\tau_{G,X}^{1(Y)} : \mathfrak{a}_{G,X}^{(Y)}/W_{G,X}^{(Y)} \rightarrow \mathfrak{g}//G$.

The morphism $\tau_{G,X}^{1(Y)}$ is, by definition, the composition of the closed embedding $\mathfrak{a}_{G,X}^{(Y)}/W_{G,X}^{(Y)} \hookrightarrow \mathfrak{l}//N_0$ and the morphism $\mathfrak{l}//N_0 \rightarrow \mathfrak{g}//G$ induced by the restriction of functions. The last morphism is finite in virtue of the Chevalley restriction theorem. Thus $\tau_{G,X}^{1(Y)}$ is finite.

Now we construct a G -invariant morphism $\widehat{\psi}_{G,X}^{(Y)} : X \rightarrow \mathfrak{a}_{G,X}^{(Y)}/W_{G,X}^{(Y)}$ such that $\psi_{G,X} = \tau_{G,X}^{1(Y)} \circ \widehat{\psi}_{G,X}^{(Y)}$. The morphism $\psi_{N_0,Y} : Y \rightarrow \mathfrak{a}_{G,X}^{(Y)}/W_{G,X}^{(Y)}$ is N_0 -equivariant. Thus we have a unique G -invariant morphism $\psi : G *_N Y \rightarrow \mathfrak{a}_{G,X}^{(Y)}/W_{G,X}^{(Y)}$ coinciding with $\psi_{N_0,Y}$ on Y . By construction, $\tau_{G,X}^{1(Y)} \circ \psi = \psi_{G,X}|_{G *_N Y}$. Since $\tau_{G,X}^{1(Y)}$ is finite, the morphism ψ can be extended to the whole variety X . This extension is denoted by $\widehat{\psi}_{G,X}^{(Y)}$.

Remark 5.2.4. The pair $(\mathfrak{a}_{G,X}^{(Y)}, W_{G,X}^{(Y)})$ depends on the choice of L, Y and so is determined uniquely up to G -conjugacy. However, the quotient $\mathfrak{a}_{G,X}^{(Y)}/W_{G,X}^{(Y)}$ and the morphisms $\widehat{\psi}_{G,X}^{(Y)}, \tau_{G,X}^{1(Y)}$ do not depend on the choice of L, Y in the following sense. Let $L' = gLg^{-1}, g \in G^\circ$, and Y' be a component of $\text{Red}_G^{N_G(L')}(X)$. There exists an element g_0 such that

$$(5.7) \quad g_0 L g_0^{-1} = L', g_0 Y = Y'.$$

Moreover, $g_{01}^{-1} g_{02} \in N_0$ for any two elements g_{01}, g_{02} satisfying (5.7). Let $g_0 \in N$ satisfy (5.7). The isomorphism $\iota : \mathfrak{a}_{G,X}^{(Y)}/W_{G,X}^{(Y)} \rightarrow \mathfrak{a}_{G,X}^{(Y')}/W_{G,X}^{(Y')}$ induced by g_0 does not depend on the choice of g_0 . Clearly, $\iota \circ \widehat{\psi}_{G,X}^{(Y)} = \widehat{\psi}_{G,X}^{(Y')}, \tau_{G,X}^{1(Y)} = \tau_{G,X}^{1(Y')} \circ \iota$.

We write $\mathfrak{a}_{G,X}/W_{G,X}, \widehat{\psi}_{G,X}, \tau_{G,X}^1$ instead of $\mathfrak{a}_{G,X}^{(Y)}/W_{G,X}^{(Y)}, \widehat{\psi}_{G,X}^{(Y)}, \tau_{G,X}^{1(Y)}$.

By the definitions of $C_{G,X}, \widetilde{\psi}_{G,X}, \tau_{G,X}$ (see the Introduction) there is a unique morphism $\tau_{G,X}^2 : C_{G,X} \rightarrow \mathfrak{a}_{G,X}/W_{G,X}$ such that $\widehat{\psi}_{G,X} = \tau_{G,X}^2 \circ \widetilde{\psi}_{G,X}, \tau_{G,X} = \tau_{G,X}^1 \circ \tau_{G,X}^2$.

In the case when X is affine our morphisms are depicted on the following commutative diagram

$$(5.8) \quad \begin{array}{ccccc} Y^{(L,L)} & \xrightarrow{\mu_{N_0/(L,L), Y^{(L,L)}} = \mu_{G,X}|_{Y^{(L,L)}}} & & & \mathfrak{a}_{G,X}^{(Y)} \\ & \searrow \cong & & & \downarrow \\ Y & \xrightarrow{\quad} & Y//\langle L, L \rangle & \xrightarrow{\quad} & Y//N_0 \rightarrow \mathfrak{a}_{G,X}/W_{G,X} \\ & \downarrow & \downarrow & \nearrow \tau_{G,X}^2 & \downarrow \tau_{G,X}^1 \\ X & \xrightarrow{\quad} & X//G & \xrightarrow{\widetilde{\psi}_{G,X} // G} & C_{G,X} \xrightarrow{\tau_{G,X}} \mathfrak{g} // G \end{array}$$

The isomorphism $Y^{(L,L)} \cong Y//\langle L, L \rangle$ takes place by Corollary 4.2.3.

5.3. The proof of Theorem 1.2.9. The general case.

Lemma 5.3.1. *Suppose X is strongly equidefectinal. Let L be a Levi subgroup of G , $N = N_G(L)$. Then the Hamiltonian N -variety $\text{Red}_G^N(X)$ is strongly equidefectinal.*

Proof. First we suppose that $X = X^{\max}$. We may assume that X is G -irreducible. X is strongly equidefectinal iff X is equidefectinal. It follows from Corollary 5.1.3 that $\text{Red}_G^N(X)$ is smooth. By (5.5), the Poisson bivector on any component of $\text{Red}_G^N(X)$ has constant rank. Proposition 5.1.7 implies that any component of $\text{Red}_G^N(X)$ is equidefectinal as a Hamiltonian L -variety. We are done.

In the general case the assertion of the lemma follows from the fact that $\text{Red}_G^N(\cdot)$ is a functor, see Proposition 5.1.6. \square

We recall that a subset $X^0 \subset X$ is said to be G -saturated if $X^0 = \pi_{G,X}^{-1}(\pi_{G,X}(X^0))$.

of Theorem 1.2.9. Let L denote the principal centralizer of X , see Definition 5.2.1. Choose an L -cross-section $Y \subset \text{Red}_G^{N_G(L)}(X)$. The natural morphism $G *_{N_G(L,Y)} Y \rightarrow X$ is an open embedding (Proposition 5.2.2) and its image is G -saturated (Proposition 5.1.2). Therefore we may assume that $X = G *_{N_G(L,Y)} Y$. By Lemma 5.3.1, Y is a strongly equidefectinal Hamiltonian L -variety. It remains to prove the theorem for the action $L : Y$. Since Y is a CN Hamiltonian L -variety, we are done (see Subsection 4.5). \square

5.4. P_u - and (P, P) -invariants. In this subsection X is an affine G -variety, G is a connected reductive group. Let P be a parabolic subgroup of G , L a Levi subgroup of P . Let P_u denote the unipotent radical of P . Put $P_0 = (P, P)$. Recall that $P_0 = P_u \ltimes (L, L)$.

The algebra $\mathbb{K}[G]^{P_u}$ is finitely generated ([Gr]). Thus $\mathbb{K}[G]^{P_0}$ is also finitely generated. Till the end of the section H denotes one of the groups P_u, P_0 . Put $\overline{G/H} = \text{Spec}(\mathbb{K}[G]^H)$.

The homogeneous space G/H is quasiffine since the character group of H is trivial. Fix an open G -equivariant embedding $G/H \hookrightarrow \overline{G/H}$.

There is the isomorphism $(\mathbb{K}[G/H] \otimes \mathbb{K}[X])^G \cong \mathbb{K}[X]^H$ induced by the restriction of functions from $(\mathbb{K}[G/H] \otimes \mathbb{K}[X])^G$ to $\{eH\} \times X \subset \overline{G/H} \times X$ (see [Pop1]). Thus the algebra $\mathbb{K}[X]^H$ is finitely generated.

The subalgebras $\mathbb{K}[X]^{P_u}, \mathbb{K}[X]^{P_0} \subset \mathbb{K}[X]$ are stable under the action of L and $(\mathbb{K}[X]^{P_u})^L = (\mathbb{K}[X]^{P_0})^L = \mathbb{K}[X]^P = \mathbb{K}[X]^G$, $(\mathbb{K}[X]^{P_u})^{(L,L)} = \mathbb{K}[X]^{P_0}$.

Put $X//H = \text{Spec}(\mathbb{K}[X]^H)$. Let $\pi_{H,X} : X \rightarrow X//H$ be the corresponding morphism. Note that this morphism is dominant but, in general, not surjective. We have a unique action $L : X//H$ such that the morphism $\pi_{H,X}$ is P_u -invariant and L -equivariant.

The morphism $\pi_{H,X}$ is the composition of the embedding $X = \{eH\} \times X \hookrightarrow \overline{G/H} \times X$ and the quotient morphism $\pi_{G, \overline{G/H} \times X}$. Thus if X^0 is an open (respectively, closed) affine G -saturated subset of X , then $X^0//H$ is identified with an open (respectively, closed) affine subvariety in $X//H$ so that $\pi_{H,X^0} = \pi_{H,X}|_{X^0}$. Note that the identification is L -equivariant.

Remark 5.4.1. There are two natural actions of L on $\overline{G/H}$. Firstly, there is the restriction of the action $G : \overline{G/H}$ to L . Secondly, there is the action $L : \overline{G/H}$ induced from the action $L : G/H$ by the right translations. This action commutes with the action of G and hence induces the action $L : X//H \cong (X \times \overline{G/H})//G$ considered above.

If otherwise is not stated, we consider the action of the first type. Note, however, that the L -orbit of eH is the same for the both actions.

In Subsection 5.5 we need to know whether there exists $\lim_{t \rightarrow 0} \tau(t)eP_0$ in $\overline{G/P_0}$, where τ is a one-parameter subgroup, $\tau : \mathbb{K}^\times \rightarrow Z(L)$.

Fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{l}$ and the corresponding root system $\Delta(\mathfrak{g})$.

Definition 5.4.2. A system of simple roots $\alpha_1, \dots, \alpha_r \in \Delta(\mathfrak{g})$ is said to be *compatible* with P , if the inclusion $\mathfrak{g}^\alpha \subset \mathfrak{p}$ is equivalent to $\alpha = \sum_{i=1}^r n_i \alpha_i$, $n_{l+1}, \dots, n_r \geq 0$.

Fix a system of simple roots $\alpha_1, \dots, \alpha_r$ compatible with P and let π_1, \dots, π_r be the corresponding system of fundamental weights.

Lemma 5.4.3. Let $\tau : \mathbb{K}^\times \rightarrow Z(L)$ be a one-parameter subgroup. Put $\xi := \frac{d}{dt} \tau(t)|_{t=0} \in \mathfrak{z}(\mathfrak{l})$. Then the limit $\lim_{t \rightarrow 0} \tau(t)P_0$ exists iff $\xi \in [\mathfrak{g}, \mathfrak{g}]$ and for all $i > l$ the inequality $\langle \pi_i, \xi \rangle \geq 0$ holds.

Proof. $\text{im } \tau \subset (G, G)$ because the limit $\lim_{t \rightarrow 0} \pi(\tau(t))$ exists in $G/(G, G)$, where π denotes the projection $G \rightarrow G/(G, G)$. Thus we may assume that G is semisimple. Replacing G with a covering and τ with a positive multiple, we may assume that G is simply connected. Let V_i be the irreducible G -module with the highest weight π_i , and $v_i \in V_i$ be a highest vector, $i = \overline{1, r}$. Put $v = v_{l+1} + \dots + v_r \in V_{l+1} \oplus \dots \oplus V_r$. There is a unique G -equivariant morphism $\overline{G/P_0} \rightarrow \overline{Gv}$ such that $eP_0 \mapsto v$. This is an isomorphism, see [PV1], Theorem 6. The limit $\lim_{t \rightarrow 0} \tau(t)v$ exists iff $\langle \xi, \pi_i \rangle \geq 0$ for all $i > l$. \square

5.5. Compatible parabolic subgroups. Let X be a normal irreducible affine Hamiltonian G -variety, L the principal centralizer and Y an L -cross-section of X , $N_0 = N_G(L, Y)$, L_0 the connected component of the inefficiency kernel of the action $L : Y // (L, L)$.

We need to choose some parabolic subgroup $P \subset G$ compatible with Y . Fix a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{l}$ and the corresponding root system $\Delta(\mathfrak{g})$.

Construction-definition of a compatible parabolic subgroup. Let us embed the $L/(L, L)$ -variety $Y^{(L, L)}$ into some $L/(L, L)$ -module V . Choose a point $y_0 \in Y^{(L, L)}$ with $(L_{y_0})^\circ = L_0$ (we recall that the L -varieties $Y^{(L, L)}$ and $Y // (L, L)$ are isomorphic, see Corollary 4.2.3). The dimension of the support S_{y_0} of y_0 (i.e. the convex hull of $L/(L, L)$ -weights of $y_0 \in V$) equals $\text{rk } L - \text{rk } L_0 = \text{def}_G(X)$ (the last equality follows from Corollary 4.2.3). To any point $\zeta \in S_{y_0}$ we assign a unique face C_ζ of a Weyl chamber of the dual root system $\Delta^\vee(\mathfrak{g})$ such that ζ is contained in the interior of C_ζ . Fix a point $\zeta \in S_{y_0}$ such that C_ζ is maximal with respect to the inclusion among all $C_{\zeta'}, \zeta' \in S_{y_0}$. Put

$$\mathfrak{q} = \mathfrak{t} \oplus \bigoplus_{\alpha, \langle \alpha^\vee, \zeta \rangle \leq 0} \mathfrak{g}^\alpha.$$

Note that \mathfrak{l} is identified with a Levi subalgebra in $\mathfrak{q}/\mathfrak{q}_u$ and that for $\alpha \in \Delta(\mathfrak{g})$

$$(5.9) \quad \langle \alpha^\vee, \zeta \rangle = 0 \Rightarrow \alpha^\vee \in \mathfrak{l}_0.$$

Indeed, if $\alpha^\vee \notin \mathfrak{l}_0$, then $\alpha^\vee y_0 \neq 0$. In other words, $S_{y_0} \not\subset \ker \alpha^\vee$. By the choice of ζ , $\langle \alpha^\vee, \zeta \rangle \neq 0$.

Choose a parabolic subalgebra $\mathfrak{p} \subset \mathfrak{q}/\mathfrak{q}_u$ such that \mathfrak{l} is a Levi subalgebra of \mathfrak{p} . Let \mathfrak{p} be the inverse image of \mathfrak{p}_0 in \mathfrak{q} under the projection $\mathfrak{q} \rightarrow \mathfrak{q}/\mathfrak{q}_u$. Let P, Q be the parabolic subgroups of G corresponding to the subalgebras $\mathfrak{p}, \mathfrak{q} \subset \mathfrak{g}$. Such a parabolic subgroup $P \subset G$ is said to be *compatible* with Y . Note that P depends on the choices of an $L/(L, L)$ -module V , an embedding $Y^{(L, L)} \hookrightarrow V$, a point y_0 , an element ζ and a subalgebra $\mathfrak{p} \subset \mathfrak{q}/\mathfrak{q}_u$.

Proposition 5.5.1. *Let P be a parabolic subgroup of G compatible with Y . The restriction of $\pi_{P_0, X}$ to $Y^{(L, L)}$ is generically finite. Moreover, for some open subset $Y^1 \subset Y^{(L, L)}$ the following condition is satisfied:*

if $y_1 \neq y_2 \in Y^1$ and $\pi_{P_0, X}(y_1) = \pi_{P_0, X}(y_2)$, then there exists $g \in (N_0 \cap G^\circ) \setminus L$ such that $y_1 = gy_2$.

The proof is based on the following lemma.

Lemma 5.5.2. *Let y_0, ζ, Q be such as in the previous construction-definition, $\tilde{L} = Z_{G^\circ}(\zeta)$, $Q_0 = (Q, Q)$. Then*

- (1) $(\tilde{L}, \tilde{L}) \cap L \subset L_0$.
- (2) $\dim L(y_0, eQ_0) = \dim L - \dim L \cap (\tilde{L}, \tilde{L})$.
- (3) $L(y_0, eQ_0)$ is closed in $Y^{(L, L)} \times \overline{G}/Q_0$.

Proof. Let us check that $L \cap (\tilde{L}, \tilde{L}) \subset L_0$. The derived subgroups of these two groups coincide. The space $\mathfrak{t} \cap [\tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}]$ is spanned by α^\vee , $\alpha \in \Delta(\tilde{\mathfrak{l}})$. By (5.9), $\mathfrak{t} \cap [\tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}] \subset \mathfrak{t} \cap \mathfrak{l}_0$ whence $\mathfrak{l} \cap [\tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}] \subset \mathfrak{l}_0$.

Note that y_0 is L_0 -invariant and that $L_{eQ_0} = L \cap (\tilde{L}, \tilde{L})$. The equality $\dim L(y_0, eQ_0) = \dim L - \dim L \cap (\tilde{L}, \tilde{L})$ follows from assertion 1.

Let $\tau : \mathbb{K}^\times \rightarrow Z(\tilde{L})$ be a one-parameter subgroup such that the limit $\lim_{t \rightarrow 0} \tau(t)(y_0, eQ_0)$ exists. Choose a system of simple roots $\alpha_1, \dots, \alpha_r \in \Delta(\mathfrak{g})$ compatible with Q (see Definition 5.4.2) and the corresponding system π_1, \dots, π_r of fundamental weights. Put $l = \text{rk}[\tilde{\mathfrak{l}}, \tilde{\mathfrak{l}}]$.

It follows from the construction of Q that $\zeta = \sum_{i>l} a_i \pi_i$ with $a_i < 0$. In other words, for any $\xi \in [\mathfrak{g}, \mathfrak{g}] \cap \mathfrak{t}$ the inequalities

$$(5.10) \quad \langle \pi_i, \xi \rangle \geq 0, \forall i > l,$$

and $\langle \xi, \zeta \rangle \geq 0$ imply $\xi \in [\tilde{l}, \tilde{l}]$.

By Lemma 5.4.3, $\xi := \frac{d}{dt}|_{t=0} \tau$ lies in $[\mathfrak{g}, \mathfrak{g}]$ and satisfies (5.10). Since the limit $\lim_{t \rightarrow 0} t y_0$ exists, $\langle \xi, \zeta \rangle \geq 0$ (let us recall that ζ is contained in the support of y_0). Thus $\xi \in [\tilde{l}, \tilde{l}]$. Since $\xi = \frac{d}{dt}|_{t=0} \tau \in \mathfrak{z}(\tilde{l})$, we get $\xi = 0$.

The Hilbert-Mumford theorem implies that the $Z(\tilde{L})$ -orbit of (y_0, eQ_0) is closed in $Y^{(L,L)} \times \overline{G/Q_0}$. Since $(\tilde{L}, \tilde{L}) \cap L$ leaves (y_0, eQ_0) invariant, the L -orbit of (y_0, eQ_0) coincides with the $Z(\tilde{L})$ -orbit. This proves assertion 3. \square

of Proposition 5.5.1. Without loss of generality we may assume that G is connected. The subset $G *__{N_0} Y \subset X$ is affine, open and G -saturated. Therefore we reduce to the case $X = G *__{N_0} Y$. In this case $X \times \overline{G/P_0} = (G *__{N_0} Y) \times \overline{G/P_0} \cong G *__{N_0} (Y \times \overline{G/P_0})$ (the last isomorphism is given by $([g, y], z) \mapsto [g, (y, g^{-1}z)], g \in G, y \in Y, z \in \overline{G/P_0}$). Clearly,

$$\pi_{N_0, Y \times \overline{G/P_0}} = \pi_{N_0/L, (Y \times \overline{G/P_0})//L} \circ \pi_{L, Y \times \overline{G/P_0}}.$$

Hence it is enough to prove that the restriction of $\pi_{L, Y \times \overline{G/P_0}}$ to $Y^{(L,L)} \times \{eP_0\}$ is generically injective. Since $Y^{(L,L)} \subset Y$ is a closed L -stable subvariety, it is enough to show the analogous claim for the morphism $\pi_{L, Y^{(L,L)} \times \overline{G/P_0}}$.

Let Q, y_0 be such as in Construction-definition above. There is a unique G -equivariant morphism $\overline{G/P_0} \rightarrow \overline{G/Q_0}$ such that $gP_0 \mapsto gQ_0$ for all $g \in G$. It is enough to check that the restriction of $\pi_{L, Y^{(L,L)} \times \overline{G/Q_0}}$ to $Y^{(L,L)} \times \{eQ_0\}$ is generically injective. It is so provided the following two claims take place:

- 1) $L_{(y_0, eQ_0)}$ is the inefficiency kernel for the action $L : \overline{L(Y^{(L,L)} \times eQ_0)}$.
- 2) The orbit $L(y_0, eQ_0)$ is closed.

The second claim is assertion 3 of Lemma 5.5.2. By assertion 1 of Lemma 5.5.2, any point of $Y^{(L,L)} \times eQ_0$ (and thence of $\overline{L(Y^{(L,L)} \times eQ_0)}$) is $L \cap (\tilde{L}, \tilde{L})$ -invariant. Assertion 2 of Lemma 5.5.2 implies the first claim. \square

5.6. P_u -reduction. The construction. We use the notation introduced in the beginning of the previous subsection.

Lemma 5.6.1. *Let \mathfrak{p} be a parabolic subalgebra in \mathfrak{g} with a Levi subalgebra \mathfrak{l} . Then the subvariety $\overline{P_u Y} \subset X$ is an irreducible component of $\mu_{G,X}^{-1}(\mathfrak{p})$.*

Proof. $\mu_{G,X}^{-1}(\mathfrak{l}^{pr} + \mathfrak{p}_u)$ is an open subvariety of $\mu_{G,X}^{-1}(\mathfrak{p})$. It follows from the definition of \mathfrak{l}^{pr} that $\mathfrak{z}_{\mathfrak{g}}(\xi) \cap \mathfrak{p}_u = \{0\}$ for any $\xi \in \mathfrak{l}^{pr}$. Therefore $\mathfrak{l}^{pr} + \mathfrak{p}_u = P_u \mathfrak{l}^{pr}$ and the action $P_u : \mathfrak{l}^{pr} + \mathfrak{p}_u$ is free. We deduce that the morphism $P_u \times \mathfrak{l}^{pr} \rightarrow \mathfrak{l}^{pr} + \mathfrak{p}_u, (g, y) \mapsto gy$, is an isomorphism. This implies that the morphism of schemes $P_u \times \mu_{G,X}^{-1}(\mathfrak{l}^{pr}) \rightarrow \mu_{G,X}^{-1}(\mathfrak{l}^{pr} + \mathfrak{p}_u), (g, y) \mapsto gy$, is an isomorphism too. Since Y is a connected (=irreducible) component of $\mu_{G,X}^{-1}(\mathfrak{l}^{pr})$, the subset $P_u Y \subset X$ is open in some component of $\mu_{G,X}^{-1}(\mathfrak{p})$. \square

Fix a parabolic subgroup $P \subset G$ compatible with Y .

Let A denote the subalgebra of $\mathbb{K}[X]$ generated by $\mathbb{K}[X]^{P_u}$ and $\{H_{\xi}, \xi \in \mathfrak{l}\}$. This is a finitely generated L -stable subalgebra in $\mathbb{K}[X]$. Denote by $\tilde{\pi}_{P_u, X}$ the morphism $X \rightarrow \text{Spec}(A)$ induced by the embedding $A \hookrightarrow \mathbb{K}[X]$, by Z an irreducible component of $\mu_{G,X}^{-1}(\mathfrak{p})$ and by I_Z

the ideal of functions from $\mathbb{K}[X]$ vanishing on Z . Note that I_Z is an L -stable ideal in $\mathbb{K}[X]$. Put $A_Z = A/(A \cap I_Z)$. This is a subalgebra of $\mathbb{K}[Z]$ consisting of the restrictions of elements from A to Z . There is the natural action of L on A_Z .

Lemma 5.6.2. (1) *For any $f, g \in A$ the restriction of $\{f, g\}|_Z$ depends only on $f|_Z, g|_Z$ and is contained in A_Z . So A_Z becomes a Poisson algebra.*
 (2) *$L : \text{Spec}(A_Z)$ is a Hamiltonian action with the hamiltonians $H_\xi|_Z, \xi \in \mathfrak{l}$.*

Proof. Firstly, we check that $\{A, I_Z\} \subset I_Z$. Note that I_Z is a minimal prime ideal of the ideal $I = \text{Span}_{\mathbb{K}[X]}(H_\xi, \xi \in \mathfrak{p}_u)$. Applying Lemma 2.1.4 to the algebra $\mathbb{K}[X]/I$ and the ideal I_Z/I , we see that it is enough to show that $\{A, I\} \subset I$. If $\eta \in \mathfrak{p}_u$, then $\{H_\xi, H_\eta\} = H_{[\xi, \eta]} \in I$ and $\{f, H_\eta\} = -\eta_* f = 0$ for $f \in \mathbb{K}[X]^{P_u}$. Since $\mathbb{K}[X]^{P_u}, H_\xi, \xi \in \mathfrak{l}$, generate A , we get $\{A, I\} \subset I$.

To prove the first assertion of the lemma it is enough to show that A is a Poisson subalgebra of $\mathbb{K}[X]$. To do this we have to check that the brackets of generators of A lie in A . Let $f, g \in \mathbb{K}[X]^{P_u}, \xi, \eta \in \mathfrak{l}$. One checks directly that $\{f, g\}, \{H_\xi, f\} \in \mathbb{K}[X]^{P_u}, \{H_\xi, H_\eta\} = H_{[\xi, \eta]}$. Assertion 2 is verified directly using Definition 3.1.1. \square

By Lemma 5.6.1, we may apply the previous construction to $Z = \overline{P_u Y}$.

Definition 5.6.3. By the P_u -reduction of X associated with Y we mean the normalization of the Hamiltonian L -variety $\text{Spec}(A_{\overline{P_u Y}})$.

Till the end of the section R denotes the P_u -reduction of X associated with Y and Z denotes the subvariety $\overline{P_u Y} \subset X$. R is equipped with the natural structure of a Hamiltonian L -variety, see Example 3.2.3. To make the notation less bulky, we write \underline{R}_Z instead $\text{Spec}(A_Z)$ and L' instead of (L, L) .

5.7. P_u -reduction. The basic properties. We preserve the notation of the previous subsection.

Let us make some remarks on morphisms between our varieties. Firstly, we have the natural dominant morphism $\tilde{\pi}_{P_u, X}|_Z : Z \rightarrow \underline{R}_Z$. This morphism is P_u -invariant and L -equivariant. The restriction of this morphism to Y is dominant and L -equivariant. Since Y is normal, this restriction can be lifted to an L -equivariant dominant morphism $\hat{\pi} : Y \rightarrow R$.

Secondly, we have the L -equivariant morphism $\nu : R \rightarrow X//P_u$ corresponding to the composition of the homomorphisms $\mathbb{K}[X]^{P_u} \twoheadrightarrow \mathbb{K}[X]^{P_u}/(I_Z \cap \mathbb{K}[X]^{P_u}) \hookrightarrow A_Z \hookrightarrow \mathbb{K}[R]$.

Lemma 5.7.1. *The following diagram is commutative. Here all horizontal arrows are quotient morphisms, the morphism $Y \rightarrow R$ is $\hat{\pi}$, $R \rightarrow \underline{R}_Z$ is the normalization, the morphism $R \rightarrow X//P_u$ coincides with ν , the morphisms $Y \rightarrow Z \rightarrow X$ are embeddings, all vertical arrows in the rectangle with the vertices $Y//L', X//P_0, X//P, Y//L$ are determined uniquely by the commutativity condition, the morphism $Z \rightarrow \underline{R}_Z$ is induced by the embedding $A_Z \hookrightarrow \mathbb{K}[Z]$, all morphisms to $\mathfrak{l}//L$ are of the form $\psi_{L, \bullet}//L$, the morphism $X//G \rightarrow \mathfrak{g}//G$ is $\mu_{G, X}//G$ and $\mathfrak{l}//L \rightarrow \mathfrak{g}//G$ is induced by the restriction of functions.*

$$\begin{array}{ccccccc}
& & Y & \longrightarrow & Y//L' & \longrightarrow & Y//L \\
& & \downarrow & & \downarrow & & \downarrow \\
& & R & \longrightarrow & R//L' & \longrightarrow & R//L \\
& & \downarrow & & \downarrow & & \downarrow \\
& & \underline{R}_Z & \longrightarrow & \underline{R}_Z//L' & \longrightarrow & \underline{R}_Z//L \\
& & \downarrow & & \downarrow & & \downarrow \\
& & X & \longrightarrow & X//P_u & \longrightarrow & X//P_\theta & \longrightarrow & X//P \cong X//G^\circ & \longrightarrow & X//G
\end{array}$$

(5.11)

Proof. The commutativity of the piece of the diagram inside the triangle with vertices $Y//L, \mathfrak{l}//L, \underline{R}_Z//L$ stems from the constructions of the moment maps for the actions of L on Y, \underline{R}_Z, R (see Subsections 5.1, 5.4). The commutativity of the piece inside the pentagon with the vertices $Z, Y, Y//L, X//G^\circ, X$ follows directly from the definitions of the L -varieties Y, \underline{R}_Z, R .

It remains to check that the piece with the vertices $\underline{R}_Z//L, \mathfrak{l}//L, X//G^\circ, \mathfrak{g}//G$ is commutative. Since the morphisms $Y \rightarrow \underline{R}_Z//L$ and $X \rightarrow X//G^\circ$ are dominant, it is enough to prove the commutativity of the rectangle with vertices $Y, X, \mathfrak{l}//L, \mathfrak{g}//G$. It is a direct consequence of $\mu_{L,Y} = \mu_{G,X}|_Y$. \square

In the sequel we suppose that all morphisms between the varieties from diagram (5.11) are the morphisms of this diagram.

Lemma 5.7.2. *The morphism $\hat{\pi}//L' : Y//L' \rightarrow R//L'$ is birational.*

Proof. The morphism $\hat{\pi}//L'$ is dominant because so is $\hat{\pi}$. Recall that $Y^{L'} \cong Y//L'$ (Corollary 4.2.3). Under this identification, $\hat{\pi}//L' = \pi_{L',R} \circ \hat{\pi}|_{Y^{L'}}$. Let us note that $\pi_{P_0,X}|_{Y^{L'}} = \pi_{L',X//P_u} \circ \nu \circ \hat{\pi}|_{Y^{L'}}$. It follows from Proposition 5.5.1 that there exists an open subset $Y^1 \subset Y^{L'} \cong Y//L'$ such that for any $y_1 \neq y_2 \in Y^1$ with $\hat{\pi}(y_1) = \hat{\pi}(y_2)$ there is $g \in N_{G^\circ}(L, Y) \setminus L$ such that $y_1 = gy_2$. But $\hat{\pi}(y_1) = \hat{\pi}(y_2)$ implies $\mu_{L,Y}(y_1) = \mu_{L,Y}(y_2)$. Since $\mu_{L,Y}(y_1)_s \subset \mathfrak{z}(\mathfrak{l}) \cap \mathfrak{l}^{pr}$, we see that $Z_{G^\circ}(\mu_{L,Y}(y_1))_s = L$. Therefore $g \in L$. Thence $\hat{\pi}(y_1) = \hat{\pi}(y_2)$ yields $y_1 = y_2$. \square

Corollary 5.7.3. $\underline{\text{def}}_L(R) = \underline{\text{def}}_G(X), \overline{\text{def}}_L(R) = \overline{\text{def}}_G(X)$.

Proof. In virtue of commutative diagram (5.11) and the fact that $\hat{\pi}$ is dominant, R is CN. Using Corollary 4.2.3, we have $\underline{\text{def}}_L(R) = \underline{\text{def}}_{Z(L)^\circ}(R//L'), \overline{\text{def}}_L(R) = \overline{\text{def}}_{Z(L)^\circ}(R//L')$. Thanks to Lemma 5.7.2, $\underline{\text{def}}_{Z(L)^\circ}(R//L') = \underline{\text{def}}_{Z(L)^\circ}(Y//L'), \overline{\text{def}}_{Z(L)^\circ}(R//L') = \overline{\text{def}}_{Z(L)^\circ}(Y//L')$. To complete the proof apply Corollary 4.2.3 to the action $L : Y$ and use Proposition 5.1.7. \square

Lemma 5.7.4. *The subalgebra $A_Z^{L'} \subset A_Z$ is generated by $f|_Z, H_\xi|_Z, f \in \mathbb{K}[X]^{P_0}, \xi \in \mathfrak{z}(\mathfrak{l})$.*

Proof. Put $J = \text{Span}_{A_Z}(H_\xi|_Z, \xi \in [\mathfrak{l}, \mathfrak{l}])$. Since \underline{R}_Z is CN, $\mu_{L',\underline{R}_Z}^{-1}(0) \subset \underline{R}_Z^{L'}$ and the restriction of π_{L',\underline{R}_Z} to $\mu_{L',\underline{R}_Z}^{-1}(0)$ is a finite bijection (Corollary 4.2.3). In particular, the natural homomorphism $A_Z^{L'} \rightarrow A_Z/J$ is an embedding. The image of this embedding coincides with $(A_Z/J)^{L'}$.

For $f \in \mathbb{K}[X]^{P_u}, \xi \in \mathfrak{z}(\mathfrak{l})$, denote by $\overline{f}, \overline{H}_\xi$ the image of f, H_ξ in A_Z/J under the natural epimorphism $A \rightarrow A_Z/J$. Clearly, $\overline{f}, \overline{H}_\xi, f \in \mathbb{K}[X]^{P_u}, \xi \in \mathfrak{z}(\mathfrak{l})$, generate A_Z/J . Note that

$\overline{H}_\xi \in (A_Z/J)^{L'}$. Let $g \in (A_Z/J)^{L'}$. There exist $g_i \in \mathbb{K}[X]^{P_u}$, $\xi_i \in \mathfrak{z}(\mathfrak{l})$ such that $g = \sum_i \overline{g_i} \overline{H}_{\xi_i}$. There is the natural epimorphism of L' -modules $\mathbb{K}[X]^{P_u} \rightarrow \mathbb{K}[X]^{P_0}$. Denote by g_i^0 the image of g_i under this epimorphism. Since g, H_{ξ_i} are L' -invariant, $g = \sum_i \overline{g_i^0} \overline{H}_{\xi_i}$. Hence the algebra $(A_Z/J)^{L'}$ is generated by $\overline{H}_\xi, \xi \in \mathfrak{z}(\mathfrak{l}), \overline{f}, f \in \mathbb{K}[X]^{P_0}$. It remains to recall that the natural map $A_Z^{L'} \rightarrow (A_Z/J)^{L'}$ is an isomorphism. \square

Lemma 5.7.5. *The morphism $\nu//L' : R//L' \rightarrow X//P_0$ is finite.*

Proof. The algebra $\mathbb{K}[R]^{L'}$ is integral over $A_Z^{L'}$. It remains to check that $A_Z^{L'}$ is integral over $\mathbb{K}[X]^{P_0}$. By Lemma 5.7.4, it is enough to show that $H_\xi|_Z, \xi \in \mathfrak{z}(\mathfrak{l})$, is integral over $\mathbb{K}[X]^{P_0}/(\mathbb{K}[X]^{P_0} \cap I_Z)$. This stems from diagram (5.11) because $H_\xi|_Z \subset \psi_{L,\underline{R}Z}^*(\mathbb{K}[\mathfrak{l}]^L)$ and $\mathbb{K}[\mathfrak{l}]^L$ is integral over $\mathbb{K}[\mathfrak{g}]^G$. \square

Since the morphism $\hat{\pi} : Y \rightarrow R$ is dominant, we can identify $\mathbb{K}[R]$ with a subalgebra of $\mathbb{K}[Y]$.

Corollary 5.7.6. *The subalgebra $\mathbb{K}[R]^{L'} \subset \mathbb{K}[Y]^{L'}$ is the integral closure of $\pi_{P_0,X}|_Y^*(\mathbb{K}[X]^{P_0})$.*

Proof. $R//L'$ is a normal variety, the morphism $Y//L' \rightarrow R//L'$ is birational (Lemma 5.7.2), the morphism $R//L' \rightarrow X//P_0$ is finite (Lemma 5.7.5). \square

5.8. Proofs of Theorems 1.2.3, 1.2.5. We preserve the notation of Subsections 5.5-5.7.

Recall that Y is a Hamiltonian N_0 -variety. Thus the morphism $Y//L \rightarrow \mathfrak{l}//L$ is N_0/L -equivariant. Further, the morphism $Y//L \rightarrow X//G$ is the composition of $\pi_{N_0/L,Y//L}$ and the open embedding $Y//N_0 \hookrightarrow X//G$.

Lemma 5.8.1. *The subalgebra $\mathbb{K}[R]^L \subset \mathbb{K}[Y]^L$ is N_0/L -stable. The morphism $\psi_{L,R//L}$ is N_0/L -equivariant, and the morphism $R//L \rightarrow X//G$ is the quotient for the action $N_0/L : R//L$.*

Proof. The subalgebras $\mathbb{K}[X]^G, \mathbb{K}[X]^{G^\circ}$ are embedded into $\mathbb{K}[Y]^L$ via the restriction of functions to Y . By Corollary 5.7.6, $\mathbb{K}[R]^{L'}$ is the integral closure of $\pi_{P_0,X}|_Y^*(\mathbb{K}[X]^{P_0})$. Therefore $\mathbb{K}[R]^L$ is the integral closure of $\mathbb{K}[X]^{G^\circ} \subset \mathbb{K}[Y]^L$. Since $\mathbb{K}[X]^{G^\circ}$ is integral over $\mathbb{K}[X]^G$, we obtain that $\mathbb{K}[R]^L$ is the integral closure of $\mathbb{K}[X]^G$ in $\mathbb{K}[Y]^L$. The subalgebra $\mathbb{K}[R]^L \subset \mathbb{K}[Y]^L$ is N_0/L -stable because $\mathbb{K}[X]^G \subset \mathbb{K}[Y]^{N_0} = (\mathbb{K}[Y]^L)^{N_0/L}$. Since $\text{Quot}(\mathbb{K}[Y]^{N_0}) = \text{Quot}(\mathbb{K}[X]^G)$, we have $\text{Quot}((\mathbb{K}[R]^L)^{N_0/L}) = \text{Quot}(\mathbb{K}[X]^G)$. Taking into account that $\mathbb{K}[X]^G$ is integrally closed in $\text{Quot}(\mathbb{K}[X]^G)$, we get $(\mathbb{K}[R]^L)^{N_0/L} = \mathbb{K}[X]^G$. Let us recall that $\hat{\pi} : Y \rightarrow R$ is dominant. Since the morphisms $\hat{\pi}//L$ and $\psi_{L,Y//L} = \psi_{L,R//L} \circ \hat{\pi}//L$ are N_0/L -equivariant, so is $\psi_{L,R//L}$. \square

We recall that $\mathfrak{a}_{G,X}^{(Y)} = \overline{\text{im } \mu_{Z(L)^\circ, Y//L'}}$. Since $\hat{\pi} : Y \rightarrow R$ is a dominant morphism commuting with the moment maps, we have $\mathfrak{a}_{G,X}^{(Y)} = \mathfrak{a}_{L,R}$.

Proposition 5.8.2. *The subalgebra $\mathbb{K}[C_{L,R}] \subset \mathbb{K}[R]^L$ is $W_{G,X}^{(Y)}$ -stable and $\mathbb{K}[C_{G,X}] = \mathbb{K}[C_{L,R}]^{W_{G,X}^{(Y)}}$. The following diagram is commutative.*

$$(5.12) \quad \begin{array}{ccccc} R//L & \xrightarrow{\quad} & C_{L,R} & \xrightarrow{\tau_{R,L}^2} & \mathfrak{a}_{G,X}^{(Y)} \\ \downarrow & & \downarrow & & \downarrow \\ X//G & \xrightarrow{\quad} & C_{G,X} & \xrightarrow{\tau_{G,X}^2} & \mathfrak{a}_{G,X}^{(Y)}/W_{G,X}^{(Y)} \\ & & \searrow & \swarrow & \\ & & \mathfrak{g}//G & & \end{array}$$

Proof. We recall that $\mathbb{K}[X]^G = (\mathbb{K}[R]^L)^{W_{G,X}^{(Y)}}$ (Lemma 5.8.1). It follows from commutative diagram (5.11) that the subalgebra $\psi_{L,R}^*(\mathbb{K}[\mathfrak{l}]^L) \subset \mathbb{K}[X]^G$ is integral over $\psi_{G,X}^*(\mathbb{K}[\mathfrak{g}]^G)$. Therefore $\mathbb{K}[C_{L,R}]$ is the integral closure of $\mathbb{K}[C_{G,X}]$ in $\mathbb{K}[R]^L$. In particular, $\mathbb{K}[C_{L,R}]$ is $W_{G,X}^{(Y)}$ -stable. Since $\mathbb{K}[C_{G,X}]$ is integrally closed in $\mathbb{K}[X]^G = (\mathbb{K}[R]^L)^{W_{G,X}^{(Y)}}$, the equality $\mathbb{K}[C_{G,X}] = \mathbb{K}[C_{L,R}]^{W_{G,X}^{(Y)}}$ holds.

Now we shall prove that the diagram (5.12) is commutative. The only non-trivial thing here is the equality $\tau_{G,X}^2 = \tau_{L,R}^2/W_{G,X}^{(Y)}$. The latter is equivalent to $\widehat{\psi}_{G,X}/G = (\widehat{\psi}_{L,R}/L)/W_{G,X}^{(Y)}$. By the definition of $\widehat{\psi}_{G,X}$ (see Subsection 5.2), $\widehat{\psi}_{G,X}|_Y = \widehat{\psi}_{N_0,Y}$. Therefore $\widehat{\psi}_{G,X}/G|_{Y//N_0} = \widehat{\psi}_{N_0,Y}/N_0$ (we recall that $Y//N_0$ is identified with an open subset of $X//G$). Equivalently, $\widehat{\psi}_{G,X}/G|_{Y//N_0} = (\widehat{\psi}_{L,Y}/L)/W_{G,X}^{(Y)}$. It remains to recall that $\widehat{\psi}_{L,Y}/L = \widehat{\psi}_{L,R}/L \circ \widehat{\pi}/L$ and that $\widehat{\pi}/L$ is a $W_{G,X}^{(Y)}$ -equivariant morphism. \square

of Theorem 1.2.3. Replacing X with its normalization, we may assume that X is normal. The codimension of any irreducible component of a fiber of $\psi_{L,R}$ in R is not less than $\underline{\text{def}}_L(R) = \underline{\text{def}}_G(X)$. Since the quotient morphism $\pi_{L,R}$ is surjective, the same is true for any irreducible component of a fiber of $\psi_{L,R}/L$. To complete the proof it remains to apply Proposition 5.8.2. \square

of Theorem 1.2.5. The morphism $\widetilde{\psi}_{G,X}/G$ is equidimensional by Theorem 1.2.3. Since $C_{G,X}$ is normal, it follows that $\widetilde{\psi}_{G,X}/G$ is an open morphism (see [Ch]). The equality $\text{im } \widetilde{\psi}_{G,X} = \text{im } \widetilde{\psi}_{G,X}/G$ holds because $\pi_{G,X}$ is surjective.

Put $Z = C_{L,R}$, $\tau = \tau_{L,R}^2 : C_{L,R} \rightarrow \mathfrak{a}_{L,R} \cong \mathfrak{a}_{G,X}^{(Y)}$. By Proposition 5.8.2, $C_{G,X} \cong Z/W_{G,X}^{(Y)}$, τ is $W_{G,X}^{(Y)}$ -equivariant and $\tau_{G,X}^2 = \tau/W_{G,X}^{(Y)}$. Applying Proposition 4.4.2 to the action $L : R$, we see that τ is étale in all points of $\text{im } \widetilde{\psi}_{L,R}$. It follows from commutative diagram (5.12) that $\text{im } \widetilde{\psi}_{L,R} = \pi_{W_{G,X}^{(Y)},Z}^{-1}(\text{im } \widetilde{\psi}_{G,X})$. \square

5.9. The proof of Theorem 1.2.7. The following proposition (at least, its first part) seems to be quite standard.

Proposition 5.9.1. *Suppose X is a generically symplectic normal affine irreducible Hamiltonian G -variety.*

- (1) *The image of the embedding $\widetilde{\psi}_{G,X}^* : \mathbb{K}(C_{G,X}) \rightarrow \mathbb{K}(X)$ coincides with the center $\mathfrak{z}(\mathbb{K}(X)^G)$ of the Poisson field $\mathbb{K}(X)^G$.*
- (2) *Under the identification $\mathbb{K}(C_{G,X}) \cong \mathfrak{z}(\mathbb{K}(X)^G)$, the equality $\mathbb{K}[\text{im } \widetilde{\psi}_{G,X}] = \mathbb{K}[X] \cap \mathfrak{z}(\mathbb{K}(X)^G)$ holds.*

Proof. We identify $\mathbb{K}(C_{G,X})$ with $\widetilde{\psi}_{G,X}^*(\mathbb{K}(C_{G,X}))$.

It follows from the definition of $C_{G,X}$ that $\mathbb{K}(C_{G,X}) \subset \mathbb{K}(X)^G$ and that $\mathbb{K}(C_{G,X})$ contains the subalgebra $\psi_{G,X}^*(\mathbb{K}[\mathfrak{g}]^G)$ and is algebraic over this subalgebra. Since $\{H_\xi, f\} = L_{\xi*}f = 0$ for all $\xi \in \mathfrak{g}$, $f \in \mathbb{K}(X)^G$, the inclusion $\psi_{G,X}^*(\mathbb{K}[\mathfrak{g}]^G) \subset \mathfrak{z}(\mathbb{K}(X)^G)$ holds. The uniqueness property for a lifting of a derivation yields $\mathbb{K}(C_{G,X}) \subset \mathfrak{z}(\mathbb{K}(X)^G)$.

Similarly to the proof of Satz 7.6 from [Kn1], one sees that $\mathfrak{z}_{\mathbb{K}(X)}(\mathbb{K}(X)^G)$ is algebraic over the subalgebra generated by $H_\xi, \xi \in \mathfrak{g}$. So $\mathfrak{z}(\mathbb{K}(X)^G) = (\mathfrak{z}_{\mathbb{K}(X)}(\mathbb{K}(X)^G))^G$ is algebraic over $\widetilde{\psi}_{G,X}^*(\mathbb{K}[\mathfrak{g}]^G)$ and thus also over $\mathbb{K}(C_{G,X})$. To prove assertion 1 of the proposition it remains

to show that $\mathbb{K}(C_{G,X})$ is algebraically closed in $\mathbb{K}(X)^G$. This follows easily from the fact that $C_{G,X}$ is integrally closed in $\mathbb{K}[X]^G$.

Proceed to assertion 2. Clearly, $\mathbb{K}[\text{im } \tilde{\psi}_{G,X}] \subset \mathbb{K}[X]$. To prove the inclusion $\mathbb{K}(C_{G,X}) \cap \mathbb{K}[X] \subset \mathbb{K}[\text{im } \tilde{\psi}_{G,X}]$ note that the pole locus of $\tilde{\psi}_{G,X}^*(f) \in \mathbb{K}(X)$ coincides with the inverse image of the pole locus of $f \in \mathbb{K}(C_{G,X})$. \square

Now we consider a special class of Hamiltonian actions. Let X be a generically symplectic Hamiltonian G -variety. One can show, compare with [V3], Ch.2, §3, Proposition 5, that the following conditions are equivalent:

- (a) $\dim X = m_G(X) + \text{def}_G(X)$.
- (b) The field $\mathbb{K}(X)^G$ is commutative with respect to the Poisson bracket.

X is called *coisotropic*, if it satisfies the equivalent conditions (a),(b). The following statement follows immediately from Proposition 5.9.1.

Corollary 5.9.2. *Preserve the conventions of the previous proposition. If X is coisotropic, then $X \rightarrow \text{im } \tilde{\psi}_{G,X}$ is the quotient morphism.*

of Theorem 1.2.7. Let us prove that $\mathfrak{a}_{G,X}^{(Y)} \subset \mathfrak{z}(\mathfrak{l})$ is a linear subspace. It follows from Lemma 3.3.6 that $0 \in \text{im } \psi_{G,X}$. Since $\psi_{G,X} = \tau_{G,X}^1 \circ \hat{\psi}_{G,X}$, we have $0 \in \text{im } \tau_{G,X}^1$. It remains to apply Proposition 4.4.1.

Let us prove that $\tilde{\psi}_{G,X}$ is surjective. Let λ_0 be such as in Lemma 3.3.6. By the same lemma, there is an action $\mathbb{K}^\times : C_{G,X}$ such that $\tilde{\psi}_{G,X}$ is \mathbb{K}^\times -equivariant and $\lim_{t \rightarrow 0} t\lambda = \lambda_0$ for all $\lambda \in C_{G,X}$. The image of $\tilde{\psi}_{G,X}$ is \mathbb{K}^\times -stable and contains λ_0 . It follows from Theorem 1.2.5 that $\text{im } \tilde{\psi}_{G,X} \subset C_{G,X}$ is an open subset. We deduce that $\tilde{\psi}_{G,X}$ is surjective. When X is generically symplectic, we apply Proposition 5.9.1 and obtain the equality $\mathbb{K}[C_{G,X}] = \mathbb{K}[X]^G \cap \mathfrak{z}(\mathbb{K}(X)^G)$.

It remains to prove that $\tau_{G,X}^2 : C_{G,X} \rightarrow \mathfrak{a}_{G,X}/W_{G,X}$ is an isomorphism. The morphism $R//L \rightarrow C_{G,X}$ from diagram (5.12) is surjective because $\tilde{\psi}_{G,X} // G$ is so. Taking into account that the morphism $\tilde{\psi}_{L,R} // L : R//L \rightarrow C_{L,R}$ is $W_{G,X}^{(Y)}$ -equivariant, we deduce from diagram (5.12) that this morphism is also surjective. Theorem 1.2.5 implies that $\tau_{L,R}^2 : C_{L,R} \rightarrow \mathfrak{a}_{G,X}^{(Y)}$ is étale. But $\tau_{L,R}^2$ is finite and thus is an isomorphism. To complete the proof it remains to apply Proposition 5.8.2. \square

5.10. An example when $W_{G,X}^{(Y)}$ is not generated by reflections. In this subsection we construct an example of a conical symplectic irreducible affine Hamiltonian G -variety X with connected G such that the image of $W_{G,X}^{(Y)}$ in $\text{GL}(\mathfrak{a}_{G,X}^{(Y)})$ is not generated by reflections. Since G is connected, the homomorphism $W_{G,X}^{(Y)} \rightarrow \text{GL}(\mathfrak{a}_{G,X}^{(Y)})$ is an embedding and we identify the group $W_{G,X}^{(Y)}$ with its image.

Put $G_0 = \text{SL}(2) \times \text{SL}(2)$. Let V_1, V_2 (resp., V'_1, V'_2) be the two-dimensional irreducible modules over the first (resp., the second) factor. Denote by γ_V the linear automorphism of $V := V_1 \oplus V_2 \oplus V'_1 \oplus V'_2$ given by the equality $\gamma_V(v_1, v_2, v'_1, v'_2) = (v_1, -v_2, v'_1, -v'_2)$, $v_i \in V_i, v'_i \in V'_i$. Put $G = \mathbb{K}^\times \times G_0$, $\tilde{X} = T^*\mathbb{K}^\times \times V$. Clearly, \tilde{X} is a symplectic affine Hamiltonian G -variety. This is a conical variety: the action $\mathbb{K}^\times : \tilde{X} \cong G *_{G_0} (\mathbb{K} \oplus V)$ is given by $t[g, (x, v)] = [g, (t^2x, tv)]$, $g \in G, t \in \mathbb{K}^\times, x \in K \cong T_1^*(\mathbb{K}^\times), v \in V$. Furthermore, \tilde{X} is coisotropic. Indeed, $m_G(\tilde{X}) = \dim G, \dim \tilde{X} = \dim G + \text{rk } G$.

Let γ_T denote the involution of $T^*\mathbb{K}^\times$ induced by the left translation by $-1 \in \mathbb{K}^\times$ and γ the involution of \tilde{X} given by $\gamma(x, y) = (\gamma_T x, \gamma_V y)$, $x \in T^*\mathbb{K}^\times$, $y \in V$. The involution γ is Hamiltonian and \mathbb{K}^\times -equivariant. Since γ has no fixed points, the variety $X = \tilde{X}/\mathbb{Z}_2$, where the non-unit element of \mathbb{Z}_2 acts as γ , is smooth. So X is a symplectic conical coisotropic Hamiltonian G -variety.

The variety \tilde{X}/G is isomorphic to \mathbb{A}^3 . The action $\mathbb{Z}_2 : \tilde{X}/G \cong \mathbb{A}^3$ is isomorphic to the linear action of \mathbb{Z}_2 by the multiplication by matrices $\text{diag}(1, \varepsilon, \varepsilon)$, $\varepsilon = \pm 1$. We deduce that $X/G \cong (\tilde{X}/G)/\mathbb{Z}_2$ is not smooth. Using Corollary 5.9.2 and Theorem 1.2.7, we see that $W_{G,X}^{(Y)}$ is not generated by reflections.

INDEX OF NOTATION

As usual, if an algebraic group is denoted by a capital Latin letter, we denote its Lie algebra by a corresponding small German letter.

$\langle \cdot, \cdot \rangle$	the pairing of elements of two dual to each other vector spaces.
$\#S$	the cardinality of a set S .
∂_v	the partial derivative in direction of a tangent vector v .
A^\times	the group of invertible elements of an algebra A .
$H^0(X, V)$	the space of global sections of a vector bundle V over a variety X .
G°	the connected component of unit of an algebraic group G .
(G, G) (resp., $[\mathfrak{g}, \mathfrak{g}]$)	the derived subgroup (resp., subalgebra) of a group G (resp., the a Lie algebra \mathfrak{g}).
$G *_H X$	the homogeneous bundle over a homogeneous space G/H with a fiber X .
G_u (resp., \mathfrak{g}_u)	the unipotent radical of an algebraic group G (resp., of an algebraic Lie algebra \mathfrak{g}).
G_x	the stabilizer of a point x under an action of G .
\mathfrak{g}_x	the annihilator of vector x in a module over a Lie algebra \mathfrak{g} .
$[g, x]$	the class of a pair (g, x) , $g \in G$, $x \in X$ in the homogeneous bundle $G *_H X$.
$f _Y$	the restriction of a map f to a subset Y .
\mathfrak{g}^α	the root subspace in a reductive Lie algebra \mathfrak{g} corresponding to $\alpha \in \Delta(\mathfrak{g})$.
$\text{im } f$	the image of a map f .
L_ξ	the Lie derivative in direction of a vector field ξ .
$m_G(X)$	the maximal dimension of an orbit for the action of an algebraic group G on a variety X .
$N_G(H)$ (resp., $N_G(\mathfrak{h})$)	the normalizer of a subgroup $H \subset G$ (resp., of a subalgebra \mathfrak{h} of \mathfrak{g}) in a group G .
$\text{Quot}(R)$	the quotient field of a domain R .
$S_A(D)$	the symmetric algebra of an A -module D .
$\text{Span}_A(S)$	the A -submodule spanned by a subset S of some A -module.
$\text{Spec}(A)$	the affine scheme corresponding to an algebra A .
U^\perp	the orthogonal complement to a subspace $U \subset V$, where V is a vector space equipped with a nondegenerate symmetric or skew-symmetric bilinear form.
$v(f)$	the skew-gradient of a rational function f on a Poisson variety.

\overline{X}	the closure of a subset of a variety with respect to the Zariski topology.
X^G	the set of G -fixed points for the action $G : X$.
$X//G$	the categorical quotient for the action of a reductive group G on an affine variety X .
X^{max}	the subset of a Poisson variety consisting of all points $x \in X^{reg}$ satisfying $\text{rk } P_x = \max_{y \in X^{reg}} \text{rk } P_y$, where P is the Poisson bivector of X .
X^{reg}	the subset of smooth points of a variety X .
$Z(G)$ (resp., $\mathfrak{z}(\mathfrak{g})$)	the center of an algebraic group G (resp., of a Lie algebra \mathfrak{g}).
$\mathfrak{z}_{\mathfrak{g}}(\mathfrak{h})$	the centralizer of a subalgebra \mathfrak{h} in a Lie algebra \mathfrak{g} .
α^{\vee}	the dual root to a root α .
$\Delta(\mathfrak{g})$	the root system of a reductive Lie algebra \mathfrak{g} .
ξ_*	the velocity vector field corresponding to an element ξ of a Lie algebra.
ξ_s	the semisimple part of an element ξ of a reductive Lie algebra
$\pi_{G,X}$	the quotient morphism $X \rightarrow X//G$ for the action $G : X$.
φ^*	the morphism of algebras of functions induced by a morphism φ of varieties.
$\varphi//G$	the morphism $X//G \rightarrow Y//G$ induced by a G -equivariant morphism $\varphi : X \rightarrow Y$.

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